COHERENCE OF DIRECT IMAGES OF THE DE RHAM COMPLEX

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Dedicated to the memory of Egbert Brieskorn (7.7.1937-19.7.2013)

Abstract. We show the coherence of the direct images of the De Rham complex relative to a flat holomorphic map with suitable boundary conditions. For this purpose, a notion of bi-dg-algebra called the Koszul-De Rham algebra is developed.

Contents

1. Introduction

In the present paper, we prove the following theorem.

Main Theorem. *Let $\Phi : Z \to S$ be a flat holomorphic map between complex manifolds.* [1] *Assume that there exists an open subset $Z' \subset Z$ with smooth boundary satisfying i) Z' contains the critical set C_Φ of Φ, ii) the closure $\bar{Z}'$ in Z is proper over S, and iii) Z' is a weak deformation retract of Z along the fibers of Φ and iv) $\partial Z'$ is transversal to all fibers $\Phi^{-1}(t)$. Then, the direct images $\mathbb{R}^k\Phi_*(\Omega^\bullet_{Z/S}, d_{Z/S})$ of the relative De Rham complex $\Omega^\bullet_{Z/S} := \Omega^\bullet_Z/\Phi^*(\Omega^1_S)\wedge\Omega^{\bullet-1}_Z$ on Z over S are $\mathcal{O}_S$-coherent modules.*

The main Theorem is well-known for a proper and/or projective morphism Φ, since 1) the ${}'E_1$-term $\mathrm{R}^q\Phi_*(\Omega^p_{Z/S})$ of the spectral sequence defining the direct image, the so called *Hodge to De Rham spectral sequence* (2.1), is already $\mathcal{O}_S$-coherent due to the proper mapping theorem of Grauert and/or Grothendieck, and 2) the differentials on the spectral sequence (induced from the relative De Rham differential $d_{Z/S}$) are $\mathcal{O}_S$-homomorphisms so that the limit of the spectral sequence is also $\mathcal{O}_S$-coherent (see [14][13]).

Therefore, our main interest is the study of the case when Φ is a non-proper morphism between open manifolds. We give a direct and down to the earth proof of the Main Theorem by introducing the notion of a *Koszul-De Rham algebra*, which seems to detect information of the singularities of the morphism Φ and to be of interest by itself (see **Step** 3). In such a non-proper mapping setting, we also remark that the result has a close connection to a general theorem for coherent $\mathcal{D}_Z$-modules which are non-characteristic on the boundary by Houzel-Schapira [12], and its generalization to elliptic systems by Schapira-Schneiders (Theorem 4.2 in [20]), since *the relative de-Rham system is an elliptic system.*

If the range S of Φ is one-dimensional, i.e. Φ is a function, and Z is a suitably small neighborhood of an isolated critical point of Φ, then the main Theorem was

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[1]We assume that a manifold is connected, paracompact, Hausdorff and, hence, metrizable.

shown by Brieskorn [2] and then by Greuel [10] (see Hamm [11] for what happens if non-isolated singularities are admitted). Namely, in the case of an isolated critical point, Φ is locally analytically equivalent to a polynomial map, and one proves the coherence by extending Φ to a projective morphism and then applying Grothendieck's coherence theorem for projective morphisms. The result was generalized by the author in [17, 18] to the complete intersection case for higher dimensional base space S, where he did not use the above mentioned algebro geometric method in [2] but used a complex analytic method developed by Forster and Knorr [7] who gave a new proof of the Grauert proper mapping theorem [8]. Recently, jointly with Changzheng Li and Si Li, the author studied in [16] morphisms Φ which may no longer be defined locally in a neighborhood of an isolated critical point but may have multiple critical points as in the Main Theorem. Then, Φ may no longer be equivalent to a polynomial map and the algebraic method in [2] seems to be no longer applicable. However the analytic method in [17] can be generalized for this new setting, as will be presented in the present paper, where we study the De Rham cohomology group by the Čech cohomology group with respect to an atlas (3.8) of relative charts due to Forster and Knorr.

In the present new setting, the morphism Φ may also no-longer neccesarily have only isolated critical points but may have higher dimensional critical sets in the fibers of Φ. For such semi-global settings, the vanishing cycles in the nearby fibers of Φ are no-longer purely middle dimensional but mixed dimensional, and the De Rham cohomology groups are no-longer pure but mixed dimensional. Then, we need to solve some topological problems. We also need to find a suitable Stein open covering of the fibration Φ in order to apply the Forster-Knorr result to the Čech complex. This is achieved in the present paper by showing an existence of some enhanced structure on the atlas of relative charts Z (*Lemma* 5.1).

The proof of the Main Theorem is divided into the following 4 steps.

Step 1. We describe two (including Hodge to De Rham) spectral sequences, describing the direct images $\mathbb{R}\Phi_*(\Omega^\bullet_{Z/S}, d_{Z/S})$ and see that the restriction from Z to Z' induces an isomorphism: $\mathbb{R}\Phi_*(\Omega^\bullet_{Z/S}, d_{Z/S}) \simeq \mathbb{R}\Phi_*(\Omega^\bullet_{Z'/S}, d_{Z'/S})$.

Step 2. For any point $t \in S$, we find a Stein open neighborhood $S^* \subset S$ such that $Z' \cap \Phi^{-1}(S^*)$ is covered by atlases of *relative charts* in the sense of Forster-Knorr, which satisfy an additional condition, called complete intersection, and which form a family of atlases parametrized by the radius r ($^\exists r^* \leq r \leq 1$) of polydiscs.

Step 3. We introduce the Koszul-De Rham algebra $\mathcal{K}^{\bullet,\star}_{D(r)\times S^*/S^*,\mathbf{f}}$ on each relative chart $D(r) \times S^*$, as a sheaf of double dg-algebras over the dg-algebra $\Omega^\bullet_{D(r)\times S^*/S^*}$ of the relative De Rham complex, which gives an $\mathcal{O}_{D(r)\times S^*}$-free "resolution" of the relative De Rham complex $(\Omega^\bullet_{Z/S}, d_{Z/S})$ up to the critical set C_Φ, where the "gap", i.e. the cohomology groups of $\mathcal{K}^{\bullet,\star}_{D(r)\times S^*/S^*,\mathbf{f}}$ w.r.t. $\star$, is given by a sequence, indexed by $s \in \mathbb{Z}_{\geq 0}$, of complexes $(\mathcal{H}^{\bullet,s}_\Phi, d_{DR})$ of coherent $\mathcal{O}_Z$-modules supported in C_Φ.

Step 4. The Čech cohomology groups of the De Rham complex $(\Omega^\bullet_{Z/S}, d_{Z/S})$ and the lifted Čech cohomology groups of the Koszul-De Rham algebra appear periodically in the first and the second terms of a long exact sequence of cohomology groups, where the second terms are coherent near $t \in S$ due to Forster-Knorr's result [7]. The third terms of the sequence, described by the complexes $\mathcal{H}^{\bullet,s}_\Phi$ in Step 3, are also coherent on S, since C_Φ is proper over S. This shows that the first terms, i.e. the direct images of the De Rham complex, is also coherent near $t \in S$.

Since the coherence is a local property on S, this completes the proof.

Remark 1. (i) A flat map Φ is an open map and defines a family of constant

$$n := \dim_{\mathbb{C}} Z - \dim_{\mathbb{C}} S$$

dimensional fibers. So, if $n = 0$, the map Φ is proper finite and hence the Main Theorem is trivial. Therefore, in the present paper, we shall assume $n > 0$.

(ii) We introduce in the present note some tools which seem to be unknown in the literature:

a) The *atlases of some special intersection nature* (*Lemma* 3.2 and 3.3) in **Step 2**,

b) The *sequence of chain complexes* $(\mathcal{H}_{\Phi}^{\bullet,s}, d_{DR})$ $(s \in \mathbb{Z}_{>0})$ of coherent $\mathcal{O}_Z$-modules supported in the critical set C_Φ of Φ in **Step 3**.

Both are essential for our purpose to give an analytic proof of the Main Theorem.

Notation. We use cohomologies of three kinds: 1. De Rham complex, 2. derived functor of direct image Φ_*, and 3. Koszul complex. According to them, when it is possible, we distinguish their indices by the following choices: 1. "$\bullet$" or "p" for $p \in \mathbb{Z}_{\geq 0}$, 2. "$*$" or "$q$" for $q \in \mathbb{Z}_{\geq 0}$, and 3. "$\star$" or "$s$" for $s \in \mathbb{Z}_{\geq 0}$, respectively.

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2. Step 1: Hodge to De Rham spectral sequence

Throughout the present paper, we keep the setting and notation of the Main Theorem. Recall that the direct image is given by the hypercohomology $\mathbb{R}^\star\Phi_*(\Omega^\bullet_{Z/S}, d_{Z/S})$ and is described by the limit of the following two spectral sequences:

$$\begin{array}{rcl} {}'E_2^{p,q} & := & \mathrm{H}^p(\mathrm{R}^q\Phi_*(\Omega^\bullet_{Z/S}), d_{Z/S}) \\ {}''E_2^{q,p} & := & \mathrm{R}^q\Phi_*(\mathrm{H}^p(\Omega^\bullet_{Z/S}, d_{Z/S})). \end{array} \tag{2.1}$$

The E_1-term ${}'E_1^{p,q} = \mathrm{R}^q\Phi_*(\Omega^p_{Z/S})$ of the first spectral sequence is sometimes called *Hodge to De Rham (or, Frölicher) spectral sequence* for the De Rham cohomology relative to Φ.

Let us consider the second spectral sequence ${}''E_2^{q,p}$ (2.1), which we shall denote also by ${}''E_2^{q,p}(Z/S)$ when we stress its dependence on the space Z/S. We first remark that $\mathrm{Supp}(\mathrm{H}^p(\Omega^\bullet_{Z/S}, d_{Z/S})) \subset C_\Phi$ for $p > 0$ (here we recall that C_Φ is the critical set of Φ so that $\Phi|_{C_\Phi}$ is a proper morphism), since the Poincaré complex $(\Omega^\bullet_{Z/S}, d_{Z/S})$ relative to Φ is exact outside the critical set of Φ. On the other hand, we have $\mathrm{H}^0(\Omega^\bullet_{Z/S}, d_{Z/S}) \simeq \Phi^{-1}\mathcal{O}_S$ (since $n > 0$). That is, $\mathrm{H}^0(\Omega^\bullet_{Z/S}, d_{Z/S})$ is constant along fibers of Φ. Therefore, we observe (cf. [21]):

Fact 1. *Let Z' be an open subset of Z satisfying 1. $C_F \subset Z'$ and 2. Z' is a deformation retract of Z along fibers of Φ. Then, the inclusion map $Z' \to Z$ induces bijection ${}''E_2^{q,p}(Z/S) \simeq {}''E_2^{q,p}(Z'/S)$ and, hence, of the hypercohomology groups $\mathbb{R}^k\Phi_*(\Omega^\bullet_{Z/S}, d_{Z/S}) \simeq \mathbb{R}^k\Phi_*(\Omega^\bullet_{Z'/S}, d_{Z'/S})$ as $\mathcal{O}_S$-module.*

[2] In [1] (Theorem 4.1), some algebra similar to the (but differently graded) Koszul-De Rham algebra in the present paper was introduced in order to calculate the Hochshild and cyclic homology of a complete intersection affine variety (similar to the complete intersection variety U in a Stein manifold W in the present paper). It should be of interest to find a relation of the present work with the Hochshild and cyclic homology. However, since the description in [1] misses the parameter space S in the present paper, the relation seems not promptly apparent.

3. Step 2: Atlas of complete intersection relative charts

We construct an atlas consisting of charts relative to the map Φ (called *relative charts* by Forster-Knorr [7]), which satisfy an additional condition called a complete intersection (*Lemma* 3.2). The atlas shall be used in **Step 4** to calculate the limit of the spectral sequence ${}'E_1^{p,q}$ by a generalization of Čech cohomology. The construction of the atlas asks the existence of a certain covering of the manifold Z of quite general nature (*Lemma* 3.3). Since the proof of the existence of such a covering is rather of technical nature and is independent of the other part of the paper, hurrying readers are suggested to skip the present section and to go to §4 after looking at definitions and results, and to come back to the proofs if necessary.

Definition. 1. ([7]) A *relative chart* for a flat family Φ is a closed embedding

$$j \ : \ U \ \longrightarrow \ D(r) \times S_U \tag{3.1}$$

where U is an open subset of Z (which may be empty), S_U is an open subset of S with $\Phi(U) \subset S_U$ and $D(r)$ is a *polycylinder of the radius* r [3] in some $\mathbb{C}^m$ $(m \in \mathbb{Z}_{\geq 0})$ such that the diagram

$$\begin{array}{ccc} U & \xrightarrow{\ j\ } & D(r) \times S_U \\ \Phi\,|_U \searrow & & \swarrow \ pr_{S_U} \\ & S_U & \end{array} \tag{3.2}$$

commutes. We sometimes call the embedding j a relative chart, for simplicity.

Definition. 2. A relative chart is called a *complete intersection* if the j-image of U is a complete intersection subvariety in $D(r) \times S_U$. That is, there exists a sequence $f_1, \cdots, f_l$ of holomorphic functions on $D(r) \times S_U$, where l is the codimension of U in $D(r) \times S_U$:

$$l := m + \dim_{\mathbb{C}} S - \dim_{\mathbb{C}} Z = m - n$$

such that j induces a natural isomorphism $j^* : \mathcal{O}_{D(r)\times S_U}/(f_1, \cdots, f_l) \simeq \mathcal{O}_U$.

Lemma 3.1. *Let $j_k : U_k \to D_k(r) \times S_k$ $(k \in K)$ be a finite system of relative charts. Then the fiber product*

$$j_K : U_K \to D_K(r) \times S_K \tag{3.3}$$

of the morphisms j_k $(k \in K)$ over $S_K := \cap_{k\in K} S_k$, where we set $D_K(r) := \prod_{k\in K} D_k(r)$, is a relative chart.

Proof. The morphism j_K is obviously a local embedding. We need to show that its image is closed. Suppose there is a sequence $z_i \in U_K$ $(i = 1, 2, \cdots)$ such that the sequence $j_K(z_i)$ converges to a point in $D_K(r) \times S_K$. Then the projection sequence $j_k(z_i)$ also converges in $D_k \times S_k$, implying that the sequence z_i converges in U_k for all $k \in K$. Then $\lim_i \{x_i\}$ belongs to $\cap_{k\in K} U_k =: U_K$ (cf. [7] Cor. 3.2). □

Definition. 3. We shall call j_K (3.3) the *intersection of relative charts* j_k $(k \in K)$.

Remark 2. Let $\dim D_k = m_k$ and $l_k = m_k - n$ for $k \in K$. Then, $j_K(U_K)$ has codimension equal to $l_K := \sum_{k\in K} m_k - n = \sum_{k\in K} l_k + (\#K - 1)n$. Even if all j_k $(k \in K)$ are complete intersections, their intersection j_K may not necessarily be a complete intersection. Therefore, the following lemma is non-trivial.

Lemma 3.2. *Let $\Phi : Z \to S$ be any flat holomorphic map. Then there exists a function $r : Z \to \mathbb{R}_{>0}$ and a relative chart $j_z : U_z(r) \to D_z(r) \times S_z$ for all $z \in Z$ and $0 < r < r(z)$ such that 1) $j_z(z)$ is independent of r and 2) $p_1 \circ j_z : U_z(r) \to D_z(r)$ is a bijection, mapping z to the center of the polycylinder of radius r. Furthermore, any finite intersection of these relative charts is complete intersection.*

Proof. We first provide the following lemma of a quite general nature.

[3] A polycylinder of radius r is by definition a domain of the form $\{(z_1, \cdots, z_m) \in \mathbb{C}^m \mid |z_i - a_i| < r \ (i = 1, \cdots, m)\}$ where $(a_1, \cdots, a_m) \in \mathbb{C}^m$ is called the *center* of the polycylinder.

Lemma 3.3. *Any complex manifold M of dimension N admits an atlas (= a collection of open charts covering M) such that, for any point of M, the union of charts containing the point is holomorphically embeddable into an open set in $\mathbb{C}^N$.* [4]

Proof. By the assumption on manifolds (see Footnote 1), M is metrizable, and let d be a metric on M. For $p \in M$ and $r \in \mathbb{R}_{\geq 0}$, let $B(p,r) := \{q \in M \mid d(p,q) < r\}$ be the ball neighborhood of a point p of radius r. We define a function on $p \in M$ by

$R(p) := \sup\{r \in \mathbb{R}_{\geq 0} \mid B(p,r)$ is holomorphically embeddable in a domain in $\mathbb{C}^N\}$.[5]

Actually, R is a positive valued continuous function on M except if it takes constant value ∞. For any fixed real number b with $0 < b < 1/3$, we show that the atlas $\{(B(p,R(p)b), \varphi_p\}_{p \in M}$, where φ_p is a holomorphic embedding of $B(p,R(p)b)$ into $\mathbb{C}^N$, has the desired property.[6] Proof: Suppose $p \in M$ belongs to the chart $B(q,R(q)b)$ centered at $q \in M$. That means $d(p,q) < R(q)b$ and then $B(p,R(q)(1-b)) \subset B(q,R(q)(1-b+b'))$ where $b' := d(p,q)/R(q) < b$ so that $1-b+b' < 1$. Hence, the ball $B(q,R(q)(1-b+b'))$ is embeddable in $\mathbb{C}^N$, and so is $B(p,R(q)(1-b))$. This implies $R(p) \geq R(q)(1-b)$. On the other hand, for any small $\varepsilon > 0$, $B(q,R(p)-d(p,q)-\varepsilon) \subset B(p,R(p)-\varepsilon)$ is embeddable in $\mathbb{C}^N$, one gets $R(q) \geq \lim_{\varepsilon \downarrow 0}(R(p)-d(p,q)-\varepsilon) = R(p)-d(p,q)$ and, hence, $(1+b)R(q) > (1+b')R(q) = R(q)+d(p,q) \geq R(p)$. Note that the chart $B(q,R(q)b)$ is contained in the ball $B(p,R(q)2b)$ of radius $R(q)2b = R(q)(1-b) - R(q)(1-3b)$. Recalling $1-3b > 0$ and inequalities $R(q)(1-b) \leq R(p)$, $R(q) > R(p)/(1+b)$, the radius is less than $R(p)1 - R(p)(1-3b)/(1+b)) = R(p)(1-(1-3b)/(1+b))$ which is a constant $(< R(p))$ independent of the point q. That is, all charts containing p are covered by the same ball $B(p,R(p)(1-(1-3b)/(1+b))$ which is embeddable in $\mathbb{C}^N$. □

We return to the proof of *Lemma* 3.2. Let $\{(B(\underline{z},R(\underline{z})b), \varphi_{\underline{z}}\}_{\underline{z} \in Z}$ be the atlas of Z described in *Lemma* 3.3 (with the additional assumption of Footnote 5). For any point $\underline{z} \in Z$, let $S_{\underline{z}}$ be a local coordinate neighborhood of $\Phi(\underline{z})$ in S.

Then, one finds easily a positive real number $r(\underline{z})$ such that for any real r with $0 < r < r(\underline{z})$, the polycylinder $D(r)$ of radius r centered at $\varphi_{\underline{z}}(\underline{z})$ is contained in the domain $\varphi_{\underline{z}}(B(\underline{z},R(\underline{z})b)) \subset \mathbb{C}^N$ and $\Phi(\varphi_{\underline{z}}^{-1}(D(r))) \subset S_z$. Then,

$$\begin{array}{cccccc} j_{\underline{z}} & : & U_{\underline{z}}(r) := \varphi_{\underline{z}}^{-1}(D(r)) & \longrightarrow & D(r) \times S_{\underline{z}} \\ & & \underline{z}' & \longmapsto & (\varphi_{\underline{z}}(\underline{z}'), \Phi(\underline{z}')) \end{array}$$

gives a family (parametrized by r) of relative chart centered at z. The codimension l of the image $j_{\underline{z}}(U_z(r))$ in $D(r) \times S_{\underline{z}}$ is equal to $m-n = N-n = \dim_{\mathbb{C}} S$. Actually, the image is determined by a system of equations:

$$\{t_i - \Phi_i \circ \varphi_{\underline{z}}^{-1} = 0\}_{i=1}^{\dim_{\mathbb{C}} S},$$

where $(t_1, \cdots, t_{\dim_{\mathbb{C}} S})$ is a local coordinate system of $S_{\underline{z}}$ and Φ_i is the ith coordinate component of the morphism Φ. Thus $j_{\underline{z}}$ is complete intersection.

Let us show that, for any finite set $K = \{(\underline{z}, r_{\underline{z}})$ of $\underline{z} \in Z$ and $0 < r_{\underline{z}} < r(\underline{z})$ such that $U_K := \cap_{(\underline{z},r_{\underline{z}}) \in K} U_{\underline{z}}(r_{\underline{z}})$ (and, hence, $S_K := \cap_{(\underline{z},r_{\underline{z}}) \in K} S_{\underline{z}}$) is non-empty, the intersection relative chart $j_K : U_K \to D_K(r) \times S_K$ is complete intersection.

Recall that j_K is given by the fiber product morphism:

$$j_K \ : \ \underline{z}' \in U_K \longmapsto ((\varphi_{\underline{z}}(\underline{z}'))_{(\underline{z},r_{\underline{z}}) \in K}, \Phi(\underline{z}')) \in \prod_{(\underline{z},r_{\underline{z}}) \in K} D_{\underline{z}}(r_{\underline{z}}) \times S_K,$$

where the codimension of $j_K(U_K)$ is equal to $l_K = \#K \cdot \dim_{\mathbb{C}} S + (\#K - 1)n$.

In case of $U_K \neq \emptyset$, the existence of a point $\underline{z}_0 \in U_K$ implies the inclusion:

$$\bigcup_{(\underline{z},r_{\underline{z}}) \in K} U_{\underline{z}}(r_{\underline{z}}) \ \subset \bigcup_{(\underline{z},r_{\underline{z}}) \in K} B(\underline{z},R(\underline{z})b) \ \subset \ B(\underline{z}_0, R(\underline{z}_0)(1-\varepsilon))$$

[4] A parallel statement obtained by replacing the terminologies: complex manifold, holomorphically and $\mathbb{C}^N$ by C^∞-manifold, diferentiably and $\mathbb{R}^N$, respectively, holds by the same proof.

[5] Here, embeddings are not necessarily isometric.

[6] For our later application, we may assume further more that φ_p is extendable to a holomorphic embedding of the ball $B(p,R(p)(1-(1-3b)/(1+b))$ into $\mathbb{C}^N$ since $3b<1$ and $1-(1-3b)/(1+b))<1$.

for $\varepsilon := (1-3b)/(1+b)$ (*Lemma* 3.2). Let $z^1, \cdots, z^N$ be the coordinates of $\mathbb{C}^N$ where the ball $B(\underline{z}_0, R(\underline{z}_0)(1-\varepsilon))$ is embedded by extending the domain of $\varphi_{\underline{z}_0}$. We also denote by $\varphi_{\underline{z}}^{-1}$ ($\underline{z} \in K$) the composition map: $D_K \times S_K \to D_{\underline{z}} \to U_{\underline{z}} \subset Z$.

Then, the image $j_K(U_K)$ is determined by the following two type of equations:

1) System equations for identifying polycylinders $D_{\underline{z}}(r_{\underline{z}})$ ($\underline{z} \in K$) with each other. That is, for each fixed j with $1 \le j \le N$, all $z_j \circ \varphi_{\underline{z}}^{-1}$ ($\underline{z} \in K$) are equal to each other. There are $(\#K-1)N = (\#K-1)(n + \dim_{\mathbb{C}} S)$ number of equations:

$$z^j \circ \varphi_{\underline{z}_0}^{-1} = z^j \circ \varphi_{\underline{z}_1}^{-1} = \cdots = z^j \circ \varphi_{\underline{z}_{k^*}}^{-1} \qquad 1 \le j \le N\ .$$

2) System equations for the graph of Φ on each polycylinder $D_{\underline{z}}(r_{\underline{z}})$ ($\underline{z} \in K$). That is, for each fixed i with $1 \le i \le \dim_{\mathbb{C}} S$, $t_i = \Phi_i \circ \varphi_{\underline{z}}^{-1}$ for all $\underline{z} \in K$. There are $\#K \cdot \dim_{\mathbb{C}} S$ number of equations. However, after the identifications in 1), we do not need all equations but only for one point $\underline{z} \in K$: $t_i = \Phi_i \circ \varphi_{\underline{z}}^{-1}$ ($1 \le i \le \dim_{\mathbb{C}} S$), that is, the number of necessary equation is equal to $\dim_{\mathbb{C}} S$.

Thus the total number of necessary equations is $(\#K-1)(n+\dim_{\mathbb{C}} S) + \dim_{\mathbb{C}} S = \#K \cdot \dim_{\mathbb{C}} S_K + (\#K-1)n = \dim_{\mathbb{C}}(D_K \times S_K) - \dim_{\mathbb{C}} U_K$, showing that the image $j_K(U_K)$ is a complete intersection subvariety of $D_K \times S_K$. It is also clear that the Jacobian of this system of defining equations has constant maximal rank. [7]

This completes the proof of *Lemma* 3.2. □

Recall the domain $Z' \subset Z$ in the Main Theorem in §1 Introduction. We assume that $\partial Z'$ in Z is smooth and transversal to all fibers $\Phi^{-1}(t)$ for all $t \in S$.

Fact 2. *For any point t of S, there exist a Stein open neighborhood S^*, a finite number of relative charts over S^**

$$j_k \ :\ U_k \ \longrightarrow\ D_k(1) \times S^*, \quad 0 \le k \le k^* \tag{3.4}$$

and a real number $0 < r^* < 1$ with the properties: for all r with $r^* \le r \le 1$, set

$$U_k(r) := j_k^{-1}(D_k(r) \times S^*) \quad \textit{and} \quad Z'(r) := \cup_{k=0}^{k^*} U_k(r). \tag{3.5}$$

Then, we have the following.

1. One has the inclusions: $Z_{S^*} := \Phi^{-1}(S^*) \supset Z'(r) \supset Z'_{S^*} := \Phi^{-1}(S^*) \cap Z'$.

2. $Z'(r)$ is retractible to Z'_{S^*} along fibers of Φ.

3. For any $K \subset \{0, \cdots, k^*\}$, the relative chart j_K is a complete intersection.

Corollary. For r with $r^* \le r \le 1$, we have $\mathcal{O}_{S^*}$-isomorphisms

$$\mathbb{R}^k \Phi_*(\Omega^\bullet_{Z_{S^*}/S^*}, d_{Z_{S^*}/S^*}) \simeq \mathbb{R}^k \Phi_*(\Omega^\bullet_{Z(r)/S^*}, d_{Z(r)/S^*}). \tag{3.6}$$

Proof. For each point $z \in \bar{Z}' \cap \Phi^{-1}(t)$, we consider a relative chart $j_z : U_z(r) \to D_z(r) \times S_z$ of *Lemma* 3.2. We consider two cases.

Case 1. $z \in Z'$: Choose any real r such that $0 < r < r(z)$ and $U_z(r) \subset Z'$.

Case 2. $z \in \partial Z'$: Choose any real r such that $0 < r < r(z)$ and $U_z(r')$ (as a manifold with corners) is transversal to $\Phi^{-1}(t)$ for all real r' with $0 < r' \le r$.

Since $\bar{Z}' \cap \Phi^{-1}(t)$ is compact, we can find a finite number of relative charts $\tilde{j}_k : \tilde{U}_k \to D_k(r_k) \times \tilde{S}_k$ ($0 \le k \le k^*$) centered at points $\underline{z}_0, \cdots, \underline{z}_{k^*}$ on $\bar{Z}' \cap \Phi^{-1}(t)$ so that the union $\cup_{k=0}^{k^*} \tilde{U}_k$ contains the compact closure $\bar{Z}' \cap \Phi^{-1}(t)$. Then, we can find a Stein open neighborhood S^* of t such that 1) its compact closure $\bar{S}^*$ is contained in $\cap_{k=0}^{k^*} S_k$, 2) $\bar{Z}' \cap \Phi^{-1}(\bar{S^*})$ is contained in $\cup_{k=0}^{k^*} \tilde{U}_k(r)$, and 3) all fibers $\Phi^{-1}(t')$ for $t' \in \bar{S}^*$ and $U_k(r')$ ($0 < r' \le r_k$) for the chart j_k whose central point z_k is on the boundary $\partial Z'$. By a suitable rescaling of the coordinate system of charts, we may assume that all radii r_k ($0 \le k \le k^*$) are equal to 1. Then, due the compactness of $\bar{S}^*$, there exists a real number r^* with $0 < r* < 1$ such that $\bar{Z}' \cap \Phi^{-1}(\bar{S^*})$ is contained in $\cup_{k=0}^{k^*} \tilde{U}_k(r')$ for all r' with $r^* \le r' \le 1$. Then, we introduce the relative chart (3.4) by setting $U_k := U_k \cap j_k^{-1}(D_k(1) \times S^*)$ and define $Z'(r)$ as in (3.5). Then, 1. is trivial by definition, 2. is a routine work, for instance due to R. Thom [21],

[7] To be precise, one need to show that any point in $D_K \times S_K$ satisfying the relations 1) and 2) is in the image of j_K. But this can be shown by a routine work so that we omit it.

and 3. is true since the system of relative charts $\{\tilde{j}_k\}_{k=0}^{k^*}$ has already this property (*Lemma* 3.2). To see (3.6), we recall the argument done in **Fact 1.** $\square$

Let us briefly describe how these relative charts shall be used in the sequel.

For any Stein open subset $S' \subset S^*$ and any real number r with $r^* \le r \le 1$, we first consider the atlas (a collection of charts)

$$\mathcal{U}(r,S') := \{(U_k(r,S') := j_k^{-1}(D_k(r)\times S'), \varphi_k)\}_{k=0}^{k^*} \tag{3.7}$$

of $Z'(r,S') := \cup_{k=0}^{k^*} U_k(r,S')$. Actually, this is a Stein open covering, since the intersection $U_K(r,S') := \cap_{k\in K} U_k(r,S')$ for any subset $K \subset \{0,\cdots,k^*\}$ is isomorphic to a closed submanifold of $D_K(r)\times S'$ and, hence, is Stein. Therefore, the $'E_1$-term of the Hodge to De Rham spectral sequence $\mathrm{H}^q(Z'(r,S'),\Omega^p_{Z/S'})$ is given by the Čech complex $(\check{\mathrm{C}}^*(\mathcal{U}(r,S'),\Omega^p_{Z/S'}),\check{\delta})$ with respect to the atlas $\mathcal{U}(r,S')$.

The atlas $\mathcal{U}(r,S')$ is lifted to an atlas of relative charts:

$$\mathfrak{U}(r,S') := \{j_k|_{U_k(r,S')} : U_k(r,S') \to D_k(r)\times S'\}_{k=0}^{k^*}. \tag{3.8}$$

In §4, we construct double dg-algebras $\mathcal{K}^{\bullet,\star}_{D_K(1)\times S'/S',\mathbf{f}}$ on $D_K(1)\times S'$ (depending on a choice of bases $\mathbf{f}$ of the defining ideal of $U_K(r,S')$ in $D_K(r)\times S'$) and a natural epimorphism $\pi : \mathcal{K}^{\bullet,\star}_{D_K(1)\times S'/S',\mathbf{f}} \to \Omega^\bullet_{U_K(r,S')}$, where the kernel of π is described by the complex $(\mathcal{H}^{\bullet,s}_\Phi)_{s>0}$ of coherent sheaves, whoes support is contained in the critical set C_Φ (we use here the complete intersection property of the relative charts). Then, in §5 we construct a "lifting" $\check{\mathrm{C}}^*(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}^{\bullet,\star}_{D(r)\times S'/S',\mathbf{f}})$ of the Čech complex (here, we need once again to "lift" the atlas $\mathfrak{U}(r,S')$ to a based lifting atlas $\widetilde{\mathfrak{U}}(r,S')$ (see *Lemma* 5.1)), whose cohomology groups induces a coherent module in a neighborhood of $t\in S^*$ due to the Forster-Knorr Lemma (see *Lemma* 5.4). Since $\mathcal{K}^{\bullet,\star}_{D_K(1)\times S'/S',\mathbf{f}}$, $\Omega^\bullet_{U_K(r,S')}$ and $(\mathcal{H}^{\bullet,s}_\Phi)_{s>0}$ form an exact triangle, we obtain also the coherence of the direct image of $\Omega^\bullet_{U_K(r,S')}$.

4. Step 3: Koszul-De Rham algebras

We introduce the key concept of the present paper, called the *Koszul-De Rham algebra*, which is a double complex of locally free sheaves over a relative chart and gives a free resolution of the relative De Rham complex $\Omega^\bullet_{U/S}$ up to C_Φ.

More precisely, we slightly generalize the relative chart (3.2) $j : U \to D(r)\times S_U$ to (4.1) $j : U \to W$,[8] and the *Koszul-De Rham-algebra*, denoted by $(\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}, d_{DR}, \partial_{\mathcal{K}})$,[9] is a sheaf on W of bi-graded $\Omega^\bullet_{W/S}$-algebras equipped with 1) the double-complex structure: De Rham operator d_{DR} and Koszul operator $\partial_{\mathcal{K}}$ and 2) a natural epimorphism: $(\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}, d_{DR}, \partial_{\mathcal{K}}) \to (\Omega^\bullet_{U/S}, d_{U/S})$. If the chart (4.1) is a complete intersection as in **Step 2**, then the morphism gives a bounded $\mathcal{O}_W$-free resolution of $\Omega^\bullet_{U/S}$ up to some "error terms" $(\mathcal{H}^{\bullet,s}_\Phi)_{s>0}$.

We first slightly generalize the concept of the relative chart (3.1), (3.2).

Definition. 4. A *based relative chart* $(j,\mathbf{f})$ is a pair of a holomorphic closed embedding $j : U \to W$ of a complex variety U into a Stein variety W with a commutative diagram over a Stein variety S:

$$\begin{tikzcd} U \arrow[rr, "j"] \arrow[dr, "\Phi_U"'] & & W \arrow[dl, "\Phi_W"] \\ & S & \end{tikzcd} \tag{4.1}$$

and a finite generator system $\mathbf{f} = \{f_1,\cdots,f_l\} \subset \Gamma(W,\mathcal{O}_W)$ of the defining ideal $\mathcal{I}_U$ of the image subvariety $j(U)$ in W (i.e. $\mathcal{I}_U := \ker(j_*j^*|_{\mathcal{O}_W}) = \sum_i \mathcal{O}_W f_i$).

[8]The generalization is done mainly for notational simplification replacing $D(r)\times S_U$ by W. In application in §5, we shall use relative charts only in the form (3.2).

[9] The notation might have better been $\mathcal{K}^{\bullet,\star}_{j,\mathbf{f}}$ than $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$.

In this setting, for $p \in \mathbb{Z}_{\geq 0}$, there is a natural epimorphism $\pi = j_* j^*|_{\Omega^p_{W/S}}$

$$\Omega^p_{W/S} \xrightarrow{\pi} j_*(\Omega^p_{U/S}) \ (\simeq \Omega^p_{U/S}) \ \to \ 0, \tag{4.2}$$ [10]

between the Kähler differentials, whose kernel, depending only on $\mathcal{I}_U$, is given by

$$\sum_{i=1}^{l} f_i \cdot \Omega^p_{W/S} + \sum_{i=1}^{l} df_i \wedge \Omega^{p-1}_{W/S}.$$

We want to construct $\mathcal{O}_W$-free resolution of this ideal generated by f_i $(1 \leq i \leq l)$ and by df_i $(1 \leq i \leq l)$. We answer this problem, up to the critical set C_Φ, by introducing the *Koszul-De Rham-algebra* $(\mathcal{K}_{W/S,\mathbf{f}}, d_{DR}, \partial_{\mathcal{K}})$. [11]

Definition. The *Koszul-De Rham-algebra* associated with the based relative chart (4.1) is a sheaf of *bi-dg-algebra*s $\mathcal{K}_{W/S,\mathbf{f}}$ over the dg-algebra $\Omega^\bullet_{W/S}$ on W equipped with two (co-)boundary operators $\partial_{\mathcal{K}},\ d_{DR}$ and with bi-degrees, describe below.

Consider a sheaf on W of *graded commutative algebra*s over the dg algebra $\Omega^\bullet_{W/S}$

$$\mathcal{K}_{W/S,\mathbf{f}} := \Omega^\bullet_{W/S}\langle \xi_1, \cdots, \xi_l \rangle [\eta_1, \cdots, \eta_l] \ / \ \mathcal{I} \tag{4.3}$$

generated by indeterminates $\xi_1, \cdots, \xi_l, \eta_1, \cdots, \eta_l$, where ξ_i's (resp. η_i's) are considered as graded commutative odd (resp. even) variables in the following sense.

i) η_i's and even degree differential forms on W are commuting with all variables,

ii) ξ_i's and odd degree differentials forms on W are anti-commuting with each other, and $\mathcal{I}$ is the both sided ideal generated by

$$\xi_i \xi_j + \xi_j \xi_i = 0 \quad \text{and} \quad \xi_i \omega + \omega \xi_i = 0 \quad \text{ for } \ 1 \leq i, j \leq l \text{ and } \omega \in \Omega^1_W. \tag{4.4}$$

We equip the algebra $\mathcal{K}_{W/S,\mathbf{f}}$ with the following 3 structures.

1. **the Koszul structure**: We define Koszul boundary operator $\partial_{\mathcal{K}}$ on $\mathcal{K}_{W/S,\mathbf{f}}$ as the $\Omega^\bullet_{W/S}$-endomorphism of the algebra defined by the relations

$$\partial_{\mathcal{K}} \xi_i = f_i, \quad \partial_{\mathcal{K}} \eta_i = -df_i \quad \text{and} \quad \partial_{\mathcal{K}} 1 = 0.$$

They automatically satisfy the relation: $\partial^2_{\mathcal{K}} = 0$.

Proof. The endomorphism $\partial_{\mathcal{K}}$ is well defined on the free algebra generated by ξ_i's and η_i's. Then, one checks that the endomorphism preserves the ideal $\mathcal{I}$ generated by relations (4.4) (since $\partial_{\mathcal{K}}(\xi_i\xi_j + \xi_j\xi_i) = f_i\xi_j - \xi_i f_j + f_j \xi_i - \xi_j f_i = 0$ and $\partial_{\mathcal{K}}(\xi_i\omega + \omega\xi_i) = f_i\omega - \omega f_i = 0$), and, hence, induces the action $\partial_{\mathcal{K}}$ on the quotient $\mathcal{K}_{W/S,\mathbf{f}}$. The relation $\partial^2_{\mathcal{K}} = 0$ follows immediately from the facts $\partial^2_{\mathcal{K}}\xi_i = \partial_{\mathcal{K}} f_i = 0$ and $\partial^2_{\mathcal{K}}\eta_i = -\partial_{\mathcal{K}} df_i = 0$. □

2. **De Rham structure**: We regard $\mathcal{K}_{W/S,\mathbf{f}}$ as De Rham complex of the Grassmann algebra $\mathcal{O}_W\langle\xi_1,\cdots,\xi_l\rangle$ where ξ_i's satisfy the first half of the Grassmann relations (4.4). Then the De Rham differential operator, denoted by d_{DR}, acting on $\mathcal{K}_{W/S,\mathbf{f}}$ is given as an extension of the classical De Rham operator $d_{W/S}$ on $\Omega^\bullet_{W/S}$ by setting

$$d_{DR} = d_{W/S} + \textstyle\sum_{j=1}^{l} \eta_j \partial_{\xi_j},$$

where ∂_{ξ_j} is the derivation of the Grassmann algebra with respect to the variable ξ_j. One, first, defines this operator as an endomorphism of the free algebra before dividing by the ideal $\mathcal{I}$. Then one check directly that the endomorphism preserves the ideal $\mathcal{I}$ (since $d_{DR}(\xi_i\xi_j + \xi_j\xi_i) = \eta_i\xi_j - \xi_i\eta_j + \eta_j\xi_i - \xi_j\eta_i = 0$ and $d_{DR}(\xi_i\omega + \omega\xi_i) = \eta_i\omega - \omega\eta_i = 0$) so that it induces the required one acting on $\mathcal{K}_{W/S,\mathbf{f}}$. The second term

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[10] For a notational simplicity, we shall sometimes confuse the sheaf $\Omega^p_{U/S}$ on U with its j-direct image $j_*(\Omega^p_{U/S})$ on W. For instance, we shall write $\Omega^p_{W/S}/\Sigma_{i=1}^l (f_i \cdot \Omega^p_{W/S} + df_i \wedge \Omega^{p-1}_{W/S}) \simeq \Omega^p_{U/S}$.

[11] Usually, Koszul resolution is defined for even elements f_i's, but here we construct a resolution for odd elements df_i's together. The interpretation to regard it as the Koszul resolution for the odd elements df_i's and to introduce the variables η_i was pointed out by M. Kapranov, to whom the author is grateful.

of d_{DR} switches odd variables ξ_i to even variables η_i. We see easily the property $(d_{DR})^2 = 0$ follows from

$$d_{DR}(\xi_j) = \eta_j,\ d_{DR}(\eta_j) = 0 \ \text{ and } \ d^2_{DR}(\mathcal{O}_S) = 0.$$

De Rham differential and Koszul differentials are anti-commuting with each other

$$\partial_{\mathcal{K}} d_{DR} + d_{DR}\partial_{\mathcal{K}} = 0$$

(since $(\partial_{\mathcal{K}} d_{DR} + d_{DR}\partial_{\mathcal{K}})\xi_i = \partial_{\mathcal{K}}\eta_i + d_{DR} f_i = -df_i + df_i = 0$ and $(\partial_{\mathcal{K}} d_{DR} + d_{DR}\partial_{\mathcal{K}})\eta_i = 0 + d_{W/S}(df_i) = 0$) so that the pair $(d_{DR}, \partial_{\mathcal{K}})$ form a double complex structure on $\mathcal{K}_{W/S,\mathbf{f}}$.[12]

3. **Bi-degree decomposition**: We give an $\mathcal{O}_W$-direct sum decomposition

$$\mathcal{K}_{W/S,\mathbf{f}} := \mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}} := \oplus_{p\in\mathbb{Z}} \oplus_{s\in\mathbb{Z}} \mathcal{K}^{p,s}_{W/S,\mathbf{f}}$$

such that

i) $\mathcal{K}^{p,0}_{W/S,\mathbf{f}} = \Omega^p_{W/S}$ $(p\in\mathbb{Z})$ and $\mathcal{K}^{p,s}_{W/S,\mathbf{f}} = 0$ for either $p<0$ or $s<0$.

ii) $\partial_{\mathcal{K}} : \mathcal{K}^{p,s}_{W/S} \to \mathcal{K}^{p,s-1}_{W/S}$ and $d_{DR} : \mathcal{K}^{p,s}_{W/S} \to \mathcal{K}^{p+1,s}_{W/S}$ $(p,s\in\mathbb{Z})$.

In order to achieve this, we introduce De Rham degree and Koszul degree on $\mathcal{K}_{W/S,\mathbf{f}}$. Namely, for a monomial of the form $\omega\Xi\mathcal{E}$ where $\omega\in\Omega^p_{W/S}$ $(p\in\mathbb{Z}_{\geq 0})$ and Ξ, $\mathcal{E}$ are monomials in $\xi_1,\cdots,\xi_l$ and $\eta_1,\cdots,\eta_l$, respectively, we define degree maps

$$\begin{array}{rcl} \deg_{DR}(\omega\Xi\mathcal{E}) & := & \text{the total degree as a differential form} \;=\; p+\deg(\mathcal{E}) \\ \deg_{\mathcal{K}}(\omega\Xi\mathcal{E}) & := & \text{the total degree of the monomial } \Xi\mathcal{E} \;=\; \deg(\Xi)+\deg(\mathcal{E}) \end{array},$$

where one should note that ξ_j's are Grasmann variables and $\deg(\Xi)$s are bounded by l, but η_i's are even variables and $\deg(\mathcal{E})$s are un-bounded. If some monomials have the same degree with respect to $\deg_{DR}$ and/or $\deg_{\mathcal{K}}$, then we also call the sum of them homogeneous of the same degree with respect to $\deg_{DR}$ and/or $\deg_{\mathcal{K}}$.

The degree maps are additive with respect to the product in $\Omega^\bullet_{W/S}\langle\xi_1,\cdots,\xi_l\rangle[\eta_1,\cdots,\eta_l]$. Since the ideal $\mathcal{I}$ is generated by bi-homogeneous elements (4.4), the bi-degrees $\deg_{DR}$ and $\deg_{\mathcal{K}}$ are induced on the quotient algebra $\mathcal{K}_{W/S,\mathbf{f}}$ (4.3). So, we set

$$\mathcal{K}^{p,s}_{W/S,\mathbf{f}} := \{\omega\in\mathcal{K}_{W/S,\mathbf{f}} \mid \omega \text{ is bi-homogeneous with } (\deg_{DR},\deg_{\mathcal{K}}) = (p,s)\}.$$

for $p,s\in\mathbb{Z}$, where $\mathcal{K}^{p,s}_{W/S,\mathbf{f}} = 0$ if either p or s is negative. Each graded piece $\mathcal{K}^{p,s}_{W/S,\mathbf{f}}$ is an $\mathcal{O}_W$-coherent module (since it is isomorphic to a direct sum of some $\Omega^\bullet_{W/S}$).

This completes the definition of Koszul-De Rham algebra. For short, we sometimes call the combination of these 3 structures the *bi-dg-algebra* structure. In the following sections A), B), C), D) and E), we describe some basic properties and applications of Koszul-De Rham algebras.

A) Functoriality.

The *functoriality of Koszul-De Rham algebra*, formulated in the following Lemma, shall play a key role in Step 4 when we construct the lifted Čech coboundary operator $\check{\delta}$ (see (5.6)).

Lemma 4.1. *A morphism $\mathbf{w} : (j' : U' \to W', \mathbf{f}') \to (j : U \to W, \mathbf{f})$ between two based charts over the same base space S is a pair (w,h) of 1) a holomorphic map $w : W' \to W$ over S, whose restriction $u := w\mid_{U'}$ induces a holomorphic map $U' \to U$ so that we have the commutative diagram:*

$$\begin{tikzcd} U' \arrow[rr, "j'"] \arrow[dr] \arrow[dd, "u"'] & & W' \arrow[dl] \arrow[dd, "w"] \\ & S & \\ U \arrow[rr, "j"'] \arrow[ur] & & W \arrow[ul] \end{tikzcd},$$

[12] That this construction of De Rham structure is the universal construction of dg-structure on $\mathcal{K}_{W/S,\mathbf{f}}$ extending that on $\Omega^\bullet_{W/S}$ was pointed out by A.Voronov, to whom the author is grateful.

and a matrix $\{h_i^k\}_{i=1,\cdots,l}^{k=1,\cdots,l'}$ *with coefficients in* $\Gamma(W',\mathcal{O}_{W'})$ *such that* $w^*(f_i) = \sum_{k=1}^{l'} h_i^k f'_k$. *Then, the correspondence*

$$\mathbf{w}^\diamond(\xi_i) := \sum_k h_i^k \xi'_k \quad \text{and} \quad \mathbf{w}^\diamond(\eta_i) := \sum_k h_i^k \eta'_k + \sum_k dh_i^k \xi'_k. \tag{4.5}$$

induces a bi-dg-algebra morphism

$$\mathbf{w}^\diamond \ : \ w^*\mathcal{K}_{W/S,\mathbf{f}} \longrightarrow \mathcal{K}_{W'/S,\mathbf{f}'} \tag{4.6}$$

over the dg-algebra morphism $w^* : w^*\Omega^\bullet_{W/S} \to \Omega^\bullet_{W'/S}$. *The morphism is functorial in the sense that for a composition* $\mathbf{w_1} \circ \mathbf{w_2}$ *of morphisms, we have*

$$(\mathbf{w_1} \circ \mathbf{w_2})^\diamond \quad = \quad \mathbf{w_2}^\diamond \circ \mathbf{w_1}^\diamond$$

Proof. The correspondence (4.1) induces a morphism between free agleblas before dividing by the ideal $\mathcal{I}$, which preserves the parities of the variabls and matches with the degree counting by $\deg_{DR}$ and $\deg_{\mathcal{K}}$. We have the correspondences

$$\xi_i\xi_j + \xi_j\xi_i \quad \mapsto \quad \sum_{k,k} h_i^k h_j^l \ (\xi'_k\xi'_l + \xi'_l\xi'_k)$$

and

$$\xi_i\omega + \omega\xi_i \quad \mapsto \quad \sum_k h_i^k \ (\xi'_k w^*(\omega) + w^*(\omega)\xi'_k)$$

so that the defining ideal $\mathcal{I}$ is preserved and the bi-graded algebra homomorphism $\mathbf{w}^\diamond$ (4.6) is well-defined.
The commutativity of $\mathbf{w}^\diamond$ with $\partial_{\mathcal{K}}$:

$$\begin{aligned} \mathbf{w}^\diamond\partial_{\mathcal{K}}(\xi_i) &= \mathbf{w}^\diamond(f_i) = \textstyle\sum_k h_i^k f'_k = \sum_k h_i^k \partial_{\mathcal{K}}\xi'_k = \partial_{\mathcal{K}}(\sum_k h_i^k\xi'_k) = \partial_K \mathbf{w}^\diamond(\xi_i), \\ \mathbf{w}^\diamond\partial_{\mathcal{K}}(\eta_i) &= \mathbf{w}^\diamond(df_i) = \textstyle\sum_k d(h_i^k f'_k) \\ &= \textstyle\sum_k (h_i^k df'_k + dh_i^k f'_k) = \sum_k \partial_{\mathcal{K}} \sum_k (h_i^k\eta'_k + dh_i^k\xi'_k) = \partial_K \mathbf{w}^\diamond(\eta_i). \end{aligned}$$

The commutativity of $\mathbf{w}^\diamond$ with d_{DR}:

$$\mathbf{w}^\diamond d_{DR}(\xi_i) \ = \ \mathbf{w}^\diamond(\eta_i) = \textstyle\sum_k w_i^k\eta'_k + \sum_k dw_i^k\xi'_k = d_{DR}(\sum_k w_i^k\xi'_k) = d_{DR}(\mathbf{w}^\diamond(\xi_i)),$$

$$\mathbf{w}^\diamond d_{DR}(\eta_i) = 0, d_{DR}\mathbf{w}^\diamond(\eta_i) = d_{DR}(\textstyle\sum_k(w_i^k\eta'_k + dw_i^k\xi'_k) = \sum_k(dw_i^k\eta'_k - dw_i^k\eta'_k) = 0.$$

The functoriality of $\mathbf{w}^\diamond$: consider a composition $(w,h) = (w_1,h_1)\circ(w_2,h_2)$ of two morphisms, where $w_1 : W_2 \to W_1$ and $w_2 : W_3 \to W_2$) and $w_1^*(f_{i,1}) = \sum_k h_{i,1}^k f_{k,2}$, $w_2^*(f_{k,2}) = \sum_l h_{k,2}^l f_{l,3}$. So, $w = w_1 \circ w_2$ and $h_i^l = \sum_k w_2^*(h_{i,1}^k)h_{k,2}^l$. Then, obviously,

$$\mathbf{w}^\diamond(\xi_{i,1}) \ = \ \textstyle\sum_l \big((\sum_k w_2^*(h_{i,1}^k)h_{k,2}^l)\xi_{l,3}\big) = \sum_k w_2^*(h_{i,1}^k\xi_{k,2}) = w_2^\diamond(w_1^\diamond(\xi_{i,1})).$$

$$\begin{aligned} \mathbf{w}^\diamond(\eta_{i,1}) &= \textstyle\sum_l \Big(\big(\sum_k w_2^*(h_{i,1}^k)h_{k,2}^l\big)\eta_{l,2} + d\big(\sum_k w_2^*(h_{i,1}^k)h_{k,2}^l\big)\xi_{l,2}\Big) \\ &= \textstyle\sum_l\sum_k w_2^*(h_{i,1}^k)\big(h_{k,2}^l\eta_{l,2} + dh_{k,2}^l\xi_{l,2}\big) + \sum_k d(w_2^*(h_{i,1}^k))h_{k,2}^l)\xi_{l,2})\big) \\ &= w_2^\diamond(\textstyle\sum_k h_{i,1}^k\eta_{k,2} + \sum_k dh_{i,1}^k\xi_{k,2}) = w_2^\diamond(w_1^\diamond(\eta_{i,1})). \end{aligned}$$

□

B) Comparison π with the De Rham complex $(\Omega^\bullet_{U/S}, d_{U/S})$.

We compare the dg-algebras $(\Omega^\bullet_{U/S}, d_{U/S})$ and $(\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}, d_{DR}, \partial_{\mathcal{K}})$. We summarize and reformulate well-known facts in terms of $\deg_{\mathcal{K}}$, $\partial_{\mathcal{K}}$ and d_{DR} as follows.

Lemma 4.2. The morphism π (4.2) *satisfies the following properties.*
1. *The morphism* π *induces an exact sequence:*

$$\mathcal{K}^{\bullet,1}_{W/S,\mathbf{f}} \xrightarrow{\partial_{\mathcal{K}}} \mathcal{K}^{\bullet,0}_{W/S,\mathbf{f}} \xrightarrow{\pi} \Omega^\bullet_{U/S} \longrightarrow 0. \tag{4.7}$$

2. *The morphism* π *commutes with De Rham differentials:*

$$\begin{array}{ccc} \mathcal{K}^{\bullet,0}_{W/S,\mathbf{f}} & \xrightarrow{\pi} & \Omega^\bullet_{U/S} \\ d_{DR}\downarrow & & \downarrow d_{U/S} \\ \mathcal{K}^{\bullet+1,0}_{W/S,\mathbf{f}} & \xrightarrow{\pi} & \Omega^{\bullet+1}_{U/S} \end{array} \tag{4.8}$$

3. *If there is a morphism* $\mathbf{w} : (j', \mathbf{f}') \to (j, \mathbf{f})$ *between two based relative charts, then the morphism* π *gives natural transformation between the functors* $\mathbf{w}^\diamond$ *and* u^*.

$$\begin{array}{ccc} w^*\mathcal{K}^{\bullet,0}_{W/S,\mathbf{f}} & \xrightarrow{\pi} & u^*\Omega^\bullet_{U/S} \\ w^\diamond \downarrow & & \downarrow u^* \\ \mathcal{K}^{\bullet,0}_{W'/S,\mathbf{f}'} & \xrightarrow{\pi'} & \Omega^\bullet_{U'/S} \end{array} \tag{4.9}$$

Proof. We have only to check the following facts.

1) The complex $(K^{\bullet,0}_{W/S,\mathbf{f}}, d_{DR})$ coincides with the De Rham complex $(\Omega^\bullet_{W/S}, d_{W/S})$.
2) The image $\partial_\mathcal{K}(\mathcal{K}^{\bullet,1}_{W/S,\mathbf{f}})$ in $\mathcal{K}^{\bullet,0}_{W/S,\mathbf{f}}$ of the $\deg_\mathcal{K} = 1$ part of the algebra is equal to the ideal generated by $f_1, \cdots, f_l$ and $df_1, \cdots, df_l$ in $(\Omega^\bullet_{W/S}, d_{W/S})$ so that as $\mathcal{O}_U$-module, they are isomorphic.
3) The relative De Rham differential $d_{U/S}$ on U is coincides with the one induced from the relative De Rham differential $d_{W/S}$ on W.
4) The morphism $w^\diamond$ on the $\deg_\mathcal{K} = 0$ part of Koszul-De Rham algebra coincides with the pull-back morphism w^* of differential forms. □

The first and the second properties of *Lemma* 4.2 means that the morphism π induces a quasi equivalence of the Koszul-De Rham double complex with the De Rham complex, and the third property 3. means the naturallity of π, which shall be used, in the next section, when we compare the Čech-triple complex with coefficients in $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$ with the Čech-double complex with coefficients in $\Omega^\bullet_{U/S}$.

C) Boundedness of the Koszul-De Rham algebra.

We discuss a certain *boundedness* of the Koszul-De Rham complexes, which is crucially used in the study of triple complex in the next section 5.

Since there are no relations mixing ξ and η, by definition, the bi-degree (p, s) term of the Koszul-De Rham algebra has the following direct sum decomposition.

$$\mathcal{K}^{p,s}_{W/S,\mathbf{f}} = \bigoplus_{a=0}^{\min\{p,s\}} \bigoplus_{\substack{\Xi \text{ is a monomimal in} \\ \xi_i\text{'s of } \deg(\Xi)=s-a.}} \bigoplus_{\substack{\mathcal{E} \text{ is a monomial in} \\ \eta_i\text{'s of } \deg(\mathcal{E})=a.}} \Omega^{p-a}_{W/S}\ \Xi\ \mathcal{E}. \tag{4.10}$$

Lemma 4.3. *The set* $\{(p, s) \in \mathbb{Z}^2 \mid \mathcal{K}^{p,s}_{W/S,\mathbf{f}} \neq 0\}$ *is contained in the strip*

$$\{(p, s) \in \mathbb{Z}^2 \mid -l \le p - s \le \dim_\mathbb{C} W\} \tag{4.11}$$

Proof. Suppose that there exists a nontrivial element $\mathcal{K}^{p,s}_{W/S,\mathbf{f}} \ni \omega\Xi\mathcal{E} \neq 0$. Then, $p - s = \deg(\omega) - \deg(\Xi)$ (recall the definition of bi-degrees), where $0 \le \deg(\omega) \le \dim_\mathbb{C} W$ and $0 \le \deg(\Xi) \le l$. This gives the bound in the formula. □

Remark 4.4. *Lemma* implies that the total Koszul-De Rham complex $\mathcal{K}^{\tilde{\bullet}}_{W/S,\mathbf{f}}$ is bounded. However each term $\bigoplus_{p-s=\tilde{\bullet}} \mathcal{K}^{p,s}_{W/S,\mathbf{f}}$ of the total complex is an infinite sum, since η_i's are even variables and the multiplication of any high power of them are non-vanishing and increases simultaneously the degrees p and s. Nevertheless, such simple repetition of same terms (in stable area) seems harmless as we shall see, in the next Step 4, that by taking a lifting of Čech cohomology groups with coefficients in Koszul-De Rham algebras, they can be truncated in (5.12).

D) $\partial_\mathcal{K}$-cohomology group of Koszul-De Rham algebra.

For each fixed $p \in \mathbb{Z}_{\ge 0}$, we study the cohomology of the bounded complex:

$$0 \to \mathcal{K}^{p,p+l}_{W/S,\mathbf{f}} \xrightarrow{\partial_\mathcal{K}} \cdots \xrightarrow{\partial_\mathcal{K}} \mathcal{K}^{p,3}_{W/S,\mathbf{f}} \xrightarrow{\partial_\mathcal{K}} \mathcal{K}^{p,2}_{W/S,\mathbf{f}} \xrightarrow{\partial_\mathcal{K}} \mathcal{K}^{p,1}_{W/S,\mathbf{f}} \xrightarrow{\partial_\mathcal{K}} \mathcal{K}^{p,0}_{W/S,\mathbf{f}} \to 0. \tag{4.12}$$

Let us first fix a notation: for $p, s \in \mathbb{Z}$, set

$$\mathcal{H}^{p,s}_{W/S,\mathbf{f}} := \mathrm{Ker}\big(\partial_\mathcal{K} : \mathcal{K}^{p,s}_{W/S,\mathbf{f}} \to \mathcal{K}^{p,s-1}_{W/S,\mathbf{f}}\big)\ /\ \partial_\mathcal{K}\big(\mathcal{K}^{p,s+1}_{W/S,\mathbf{f}}\big), \tag{4.13}$$

and call it *Koszul-cohomology*, or $\partial_\mathcal{K}$*-cohomology*.

We first recall some functorial properties of them.

Lemma 4.5. i) *The De Rham operator d_{DR} on the Koszul-De Rham algebra induces*

$$d_{DR} \ : \ \ \mathcal{H}^{p,s}_{W/S,\mathbf{f}} \ \longrightarrow \ \mathcal{H}^{p+1,s}_{W/S,\mathbf{f}}$$

such that $d_{DR}^2 = 0$, which we shall call the De Rham operator on $\partial_{\mathcal{K}}$-cohomology.

ii) *Let $\mathbf{w} = (w, h) : (j', \mathbf{f}') \to (j, \mathbf{f})$ be a morphism between based relative charts, and set $u := w|'_U : U' \to U$. Then, the morphism $\mathbf{w}^\diamond$ (4.6) induces a morphism*

$$u^\diamond \ : \ \ \mathcal{H}^{p,s}_{W/S,\mathbf{f}} \ \longrightarrow \ \mathcal{H}^{p,s}_{W'/S',\mathbf{f}'}$$

which commutes with the De Rham operators on $\partial_{\mathcal{K}}$-cohomologies, and has the functorial property: $(u_1 \circ u_2)^\diamond = u_2^\diamond \circ u_1^\diamond$.

Proof. All these facts are immediate consequences of the fact that $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$ is a double complex with respect to ∂_K and d_{DR} shown in §4, and the fact that $\mathbf{w}^\diamond$ is a bi-dg-algebra homomorphism from $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$ to $\mathcal{K}^{\bullet,\star}_{W'/S,\mathbf{f}'}$ (*Lemma* 4.1). $\square$

Note that the ∂_K-cohomology groups are $\mathcal{O}_W$-coherent modules, since the modules $\mathcal{K}^{p,s}_{W/S,\mathbf{f}}$ are $\mathcal{O}_W$-coherent and the $\partial_{\mathcal{K}}$ are $\mathcal{O}_W$-homomorphisms ((4.10)). We analyze the $\partial_{\mathcal{K}}$-cohomologies in that context. The first basic fact is that they are defined on U.

Lemma 4.6. *The $\partial_{\mathcal{K}}$-cohomology $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ ($p, s \in \mathbb{Z}$) is an $\mathcal{O}_U$-coherent module.*

Proof. Recall that the defining ideal of U is given by $\mathcal{I}_U = \sum_i \mathcal{O}_W f_i$. Therefore, to be an $\mathcal{O}_U$-module, we have only to show that $f_i \mathcal{H}^{p,s}_{W/S,\mathbf{f}} = 0$. Let $\omega \in \mathcal{K}^{p,s}_{W/S,\mathbf{f}}$ be a presentative of an element $[\omega] \in \mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ such that $\partial_{\mathcal{K}}\omega = 0$. Then we calculate that $\partial_{\mathcal{K}}(\xi_i\omega) = \partial_{\mathcal{K}}(\xi_i)\omega + \xi_i\partial_{\mathcal{K}}\omega = f_i\omega$. That is, the class $[f_i\omega] \in \mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ is equal to zero. $\square$

We know already by (4.7) that the zero-th $\partial_{\mathcal{K}}$-cohomology is naturally given by

$$\mathcal{H}^{p,0}_{W/S,\mathbf{f}} \quad \simeq \quad \Omega^p_{U/S} \tag{4.14}$$

which is compatible with the De Rham operator action.

In order to analyze the support of $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ more carefully, recall the direct sum expression of the Koszul-De Rham algebra (4.10). We observe that the Koszul boundary operator $\partial_{\mathcal{K}}$ splits into a sum $\partial + \tilde{\partial}$, where each ∂ and $\tilde{\partial}$ is defined as $\Omega^\bullet_{W/S}$-linear endomorphisms such that

$$\partial\xi_i = f_i \ , \ \partial\eta_i = 0 \ , \ \partial 1 = 0 \qquad \text{and} \qquad \tilde{\partial}\eta_i = df_i \ , \ \tilde{\partial}\xi_i = 0 \ , \ \tilde{\partial}1 = 0 \ .$$

We see immediately the relations $\partial, \tilde{\partial} : \mathcal{K}^{p,s}_{W/S,\mathbf{f}} \to \mathcal{K}^{p,s-1}_{W/S,\mathbf{f}}$ for all $p, s \in \mathbb{Z}$ and $\partial^2 = \tilde{\partial}^2 = \partial\tilde{\partial} + \tilde{\partial}\partial = 0$. That is, for each fixed $p \in \mathbb{Z}$, the subcomplex $(\mathcal{K}^{p,\star}_{W/S,\mathbf{f}}, \partial_{\mathcal{K}})$ can be regarded as the total complex of a double complex $(\mathcal{K}^{p,\star}_{W/S,\mathbf{f}}, \partial, \tilde{\partial})$.

More precisely, let us denote by $\Omega^a_{W/S}\xi^b\eta^c$ the space spanned by those elements of the form $\omega\Xi\mathcal{E}$ with $\omega \in \Omega^a_{W/S}$, and Ξ and $\mathcal{E}$ are monomials of ξ_j's and η_j's of degree $\deg(\Xi) = b$ and $\deg(\mathcal{E}) = c$, respectively. Then, we have $\partial : \Omega^a_{W/S}\xi^b\eta^c \to \Omega^a_{W/S}\xi^{b-1}\eta^c$ and $\tilde{\partial} : \Omega^a_{W/S}\xi^b\eta^c \to \Omega^{a+1}_{W/S}\xi^b\eta^{c-1}$. So, by putting $\mathcal{K}^{p,\{b,c\}}_{W/S,\mathbf{f}} := \Omega^{p-c}_{W/S}\xi^b\eta^c$ for $p, b, c \in \mathbb{Z}$, we get double complex $(\mathcal{K}^{p,\{\star,\tilde{\star}\}}_{W/S,\mathbf{f}}, \partial, \tilde{\partial})$ (where $\mathcal{K}^{p,\{b,c\}}_{W/S,\mathbf{f}} \neq 0$ only when $0 \le b \le l$ and $0 \le c \le p$), and we have the identification of the total complex with the original Koszul complex:

$$(\oplus_{b+c=\star}\mathcal{K}^{p,\{b,c\}}_{W/S,\mathbf{f}}, \partial + \tilde{\partial}) \quad = \quad (\mathcal{K}^{p,\star}_{W/S,\mathbf{f}}, \partial_{\mathcal{K}}) \tag{4.15}$$

for each fixed $p \in \mathbb{Z}$. Explicitly, the double complex is given in the following Table.

$$\begin{array}{ccccccccccc}
\Omega^p_{W/S} & \overset{\tilde\partial}{\leftarrow} & \Omega^{p-1}_{W/S}\eta^1 & \overset{\tilde\partial}{\leftarrow} & \cdots & \overset{\tilde\partial}{\leftarrow} & \Omega^1_{W/S}\eta^{p-1} & \overset{\tilde\partial}{\leftarrow} & \mathcal{O}_W\eta^p & \leftarrow 0 \\
\uparrow\partial & & \uparrow\partial & & & & \uparrow\partial & & \uparrow\partial & \\
\Omega^p_{W/S}\xi^1 & \overset{\tilde\partial}{\leftarrow} & \Omega^{p-1}_{W/S}\xi^1\eta^1 & \overset{\tilde\partial}{\leftarrow} & \cdots & \overset{\tilde\partial}{\leftarrow} & \Omega^1_{W/S}\xi^1\eta^{p-1} & \overset{\tilde\partial}{\leftarrow} & \mathcal{O}_W\xi^1\eta^p & \leftarrow 0 \\
\uparrow\partial & & \uparrow\partial & & & & \uparrow\partial & & \uparrow\partial & \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & \\
\uparrow\partial & & \uparrow\partial & & & & \uparrow\partial & & \uparrow\partial & \\
\Omega^p_{W/S}\xi^{l-1} & \overset{\tilde\partial}{\leftarrow} & \Omega^{p-1}_{W/S}\xi^{l-1}\eta^1 & \overset{\tilde\partial}{\leftarrow} & \cdots & \overset{\tilde\partial}{\leftarrow} & \Omega^1_{W/S}\xi^{l-1}\eta^{p-1} & \overset{\tilde\partial}{\leftarrow} & \mathcal{O}_W\xi^{l-1}\eta^p & \leftarrow 0 \\
\uparrow\partial & & \uparrow\partial & & & & \uparrow\partial & & \uparrow\partial & \\
\Omega^p_{W/S}\xi^l & \overset{\tilde\partial}{\leftarrow} & \Omega^{p-1}_{W/S}\xi^l\eta^1 & \overset{\tilde\partial}{\leftarrow} & \cdots & \overset{\tilde\partial}{\leftarrow} & \Omega^1_{W/S}\xi^l\eta^{p-1} & \overset{\tilde\partial}{\leftarrow} & \mathcal{O}_W\xi^l\eta^p & \leftarrow 0 \\
\uparrow & & \uparrow & & & & \uparrow & & \uparrow & \\
0 & & 0 & & & & 0 & & 0 &
\end{array}$$

Table: Double complex $(\mathcal{K}^{p,\{\star,\tilde\star\}}_{W/S,\mathbf{f}}, \partial, \tilde\partial)$

Definition. 5. A based relative chart $(j, \mathbf{f})$ is called a *complete intersection* if its underlying relative chart (4.1) satisfies the following 1), 2) and 3).

1) The varieties U, W and S are smooth.
2) The map $\Phi_W : W \to S$ is submersive, in particular, Φ_W has no critical points.
3) The U is a complete intersection subvariety of W and $f_1, \cdots, f_l$ is a minimal system of equations for U, i.e. $f_1, \cdots, f_l$ form a regular sequence on W.

From now on through the end of the present paper, we study only based relative charts which is complete intersection. For such relative chart, we say that the morphism $\Phi_U : U \to S$ is critical at a point in U if Φ_U is not submersive at the point. That is, the variety of critical set is given by

$$C_{\Phi_U} := \{x \in U \mid \text{the rank of the Jacobian of } \Phi_U \text{ at } x \text{ is less than } \dim_{\mathbb{C}} S\}, \tag{4.16}$$

whose defining ideal $\mathcal{I}_{C_{\Phi_U}}$ in $\mathcal{O}_U$ is the one generated by the minors of size $\dim_{\mathbb{C}} S$ of the Jacobian matrix of Φ_U. We now prove some basic properties of $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ which we shall use in the next section seriously.

Lemma 4.7. Suppose that a based relative chart (4.1) *is a complete intersection. Then, we have*
(1) *the $\mathcal{O}_U$-module $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ for $s, p \in \mathbb{Z}$ together with the action of De Rham operator d_{DR} is independent of the choice of basis $\mathbf{f}$ but depends only on the morphism Φ_U,*
(2) *the support of the module $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ for $s > 0$ is contained in the critical set C_{Φ_U}.*

Proof. Before we start to prove this Lemma, we visit the double complex $\mathcal{K}^{p,\{b,c\}}_{W/S,\mathbf{f}}$ given in **Table** under the complete intersection assumption.

By **Definition** 1) and 2) of complete intersection, $\Omega^{\bullet}_{W/S}$ is an $\mathcal{O}_W$-locally free modules of finite rank. The ith vertical direction (w.r.t. the coboundary operator ∂) of the diagram for $i = 0, 1, \cdots, p$ is the classical Koszul complex on the locally free module $\Omega^{p-i}_{W/S}$ for the regular sequence $f_1, \cdots, f_l$ (recall that ξ_i's are odd variables), which is exact except at the zeroth stage, and the cokernel module at the zeroth stage is an $\mathcal{O}_U$ locally free module isomorphic to $\Omega^{p-i}_{W/S}\eta^i/(f_1, \cdots, f_l)\Omega^{p-i}_{W/S}\eta^i = \big(\overset{p-i}{\wedge}\Omega^1_{W/S}\eta^i\big)\otimes_{\mathcal{O}_W}\mathcal{O}_U$. Between the modules, $\tilde\partial$ induces a cochain complex structure, denoted again by $\tilde\partial$. In view of (4.15), this chain complex

$$\Big(\big(\overset{p-\star}{\wedge}\Omega^1_{W/S}\big)\eta^\star\otimes_{\mathcal{O}_W}\mathcal{O}_U,\ \tilde\partial\,\Big) \tag{$**$}$$

is quasi-isomorphic to the Koszul complex $(\mathcal{K}^{p,\star}_{W/s,\mathbf{f}}, \partial_{\mathcal{K}})$. Therefore, we show that the cohomology groups of (**) does not depend on the choice of the bases $\mathbf{f}$.

We provide, now, the following elementary but quite useful reduction lemma.

Lemma 4.8. *Let f_1 be the first element of the basis $\mathbf{f} = \{f_1, \cdots, f_l\}$. Suppose df_1 is a part of $\mathcal{O}_W$-free basis of the module $\Omega^1_{W/S}$. Consider the hypersurface $W' := \{f_1 = 0\} \subset W$ and set $\mathbf{f}' = \{f_2, \cdots, f_l\}$. Then, we have*
(1) $(j' : U \to W', \mathbf{f}')$ is also a complete intersection based relative chart,
(2) the inclusion map $\iota : W' \subset W$ together with the correspondence $\eta_1 \mapsto 0$ induces a morphism between the based relative charts and a quasi-isomorphism of chain complexes of $\mathcal{O}_U$-modules:

$$\big(\big(\overset{p-\star}{\wedge}\Omega^1_{W/S}\big)\eta^\star \otimes_{\mathcal{O}_W} \mathcal{O}_U, \tilde{\partial}\big) \to \big(\big(\overset{p-\star}{\wedge}\Omega^1_{W'/S}\big)\eta'^\star \otimes_{\mathcal{O}_{W'}} \mathcal{O}_U, \tilde{\partial}\big).$$

(3) The $\mathcal{O}_U$-isomorphism: $\mathcal{H}^{p,s}_{W/S,\mathbf{f}} \simeq \mathcal{H}^{p,s}_{W'/S,\mathbf{f}'}$ $(p, s \in \mathbb{Z})$ obtained from this quasi-isomorphism coincides with $(\iota|_U)^\diamond$ (recall Lemma 4.5 ii)*). In particular, the isomorphism commutes with the De Rham operator action.*

Proof. (1) The fact that df_1 is a part of $\mathcal{O}_W$-free basis $\Omega^1_{W/S}$ implies that W' is a smooth variety and that the restriction $\Phi'_U := \Phi_U|_{W'}$ is still submersive.

(2) On the space W, the two chain complexes of sheaves $\big(\big(\overset{p-\star}{\wedge}\Omega^1_{W'/S}\big)\eta'^\star \otimes_{\mathcal{O}_{W'}} \mathcal{O}_U, \tilde{\partial}\big)$ and $\big(\big(\overset{p-\star}{\wedge}\Omega^1_{W/S}/\mathcal{O}_W df_1\big)\eta'^\star \otimes_{\mathcal{O}_W} \mathcal{O}_U, \tilde{\partial}\big)$ are naturally isomorphic, since $\mathcal{O}_U \simeq \mathcal{O}_W/(f_1, \cdots, f_l) \simeq \mathcal{O}_{W'}/(f_2, \cdots, f_l)$. Therefore, in order to show (2), it is sufficient to show the following general linear algebraic facts (cf. [19]).

Proposition. Let M be a free module of finite rank over a noetherian commutative unitary ring R. Let $\wedge^* M$ be the Grassmann algebra of M over R. For given elements $\omega_1, \cdots, \omega_k$ of M, consider the polynomial ring $\wedge^* M[\eta]$ of k variables $\eta_1, \cdots, \eta_k$ equipped with a Koszul differential $\tilde{\partial}$ defined by setting $\tilde{\partial}(\eta_i) = \omega_i$ on it.
(a) Let $\mathfrak{a}$ be the ideal in R generated by the coefficients of $\omega_1 \wedge \cdots \wedge \omega_k$. Then, i-th cohomology group of $(\wedge^* M[\eta], \tilde{\partial})$ vanishes for $i <$ depth$(\mathfrak{a})$.
(b) If ω_1 is a part of some R-free basis system of M, then the natural chain morphism $(\wedge^* M[\eta], \tilde{\partial}) \to (\wedge^* M/R\omega_1[\eta'], \tilde{\partial}')$ (where η' is the indeterminates $\eta_2, \cdots, \eta_k$ such that $\tilde{\partial}'(\eta_i) = \omega_i$ $(i = 2, \cdots, k)$ and η_1 is mapped to 0) is quasi isomorphic.

(3) First, we note that there is a slight abuse of notation. Namely, we have needed to fix the coefficient matrix h of the transformation $\iota^*(f_1) = 0$ and $\iota^*(f_i) = f_i$ for $i = 2, \cdots, l$ in order that $(\iota, h)^\diamond : \mathcal{K}^{p,s}_{W/S,\mathbf{f}} \to \mathcal{K}^{p,s}_{W'/S,\mathbf{f}'}$ is defined (recall *Lemma* 4.1). Once $(\iota, h)^\diamond$ is defined in this manner, then we have $(\iota, h)^\diamond(\xi_1) = (\iota, h)^\diamond(\eta_1) = 0$ and $(\iota, h)^\diamond(\xi_i) = \xi_i$, $(\iota, h)^\diamond(\eta_i) = \eta_i$ for $i = 2, \cdots, l$. Then, we observe that $(\iota, h)^\diamond$ is compatible with the double complex $\mathcal{K}^{p,b,c}_{W/S,\mathbf{f}}$ decomposition, inducing morphism $(\iota, h)^{\diamond, double} : \mathcal{K}^{p,b,c}_{W/S,\mathbf{f}} \to \mathcal{K}^{p,b,c}_{W'/S,\mathbf{f}'}$ for all $p, b, c \in \mathbb{Z}$. Then, the chain map in (2) obviously coincides with the one induced from $(\iota, h)^{\diamond, double}$. □

Let us come back to the proof of *Lemma* 4.7.

Proof of Lemma 4.7 (1).

Suppose that there are two complete intersection based charts $(j^1 : U_1 \to W_1, \mathbf{f}^1)$ and $(j^2 : U_2 \to W_2, \mathbf{f}^2)$ over the same base set S and points $z_1 \in U_1$ and $z_2 \in U_2$ such that there is a local bi-holomorphic map $(U^1, z_1) \simeq (U^2, z_2)$ which commutes with the maps Φ_{U^1} and Φ_{U^2} in neighborhoods of z_1 and z_2. Then we show that there is a natural $\mathcal{O}_{U_1,z_1}$-$\mathcal{O}_{U_2,z_2}$-isomorphism of the stalks:

$$\mathcal{H}^{p,s}_{W^1/S,\mathbf{f^1},z_1} \simeq \mathcal{H}^{p,s}_{W^2/S,\mathbf{f^2},z_2}$$

which is equivariant with the De-Rham actions. By shrinking the relative charts j^i $(i = 1, 2)$ suitably, we may assume $U_1 \simeq U_2$, and, further more, that W_i is a Stein domain of $U_i \times \mathbb{C}^{l_i}$ such that i) the embedding j^i is realized by the isomorphism

$U_i \simeq U_i \times 0 \subset W_i \subset U_i \times \mathbb{C}^{l_i}$ and ii) the composition of the embedding of W_i in $U_i \times \mathbb{C}^{l_i}$ with the projection to the j-th component of $\mathbb{C}^{l_i}$ is equal to the j-th component, say f^i_j, of $\mathbf{f}^i$ $(i=1,2)$ (however, the compositions of the embedding $W_i \to U_i \times \mathbb{C}^{l_i}$ with the projection to U_i and with Φ_{U_i} may not necessarily coincide with the morphism $\Phi_{W_i} : W_i \to S$).

The proof is achieved by introducing an auxiliarly third based relative chart (j, W). Namely, set $U := U^1 \simeq U^2$ and let $z \in U$ be the point corresponding to $z_i \in U_i$. Then, $W := W_1 \times_U W_2$ may naturally considered as a Stein domain in $U \times \mathbb{C}^{l_1+l_2}$ such that $W_i = (U \times \mathbb{C}^{l_i}) \cap W$ $(i = 1,2)$. Since W is Stein and the maps $\Phi_{W_1} : W_1 \to S$ and $\Phi_{W_2} : W_2 \to S$ coincide with Φ_U on the intersection $W^1 \cap W^2 = U$, we can find a holomorphic map $\Phi_W : W \to S$ (up to some ambiguity) which coincides with Φ_{W_i} on each W_i (e.g. $p^*_{W_1}\Phi_{W_1} + p^*_{W_2}\Phi_{W_2} - p^*_U\Phi_U$). We shall denote again by f^1_j (resp. f^2_j) the j-th (resp. l_1+j-th) component of the coordinate of $\mathbb{C}^{l_1+l_2}$.Then, $\mathbf{f} := \mathbf{f}_1 \cup \mathbf{f}_2$ forms a basis of the defining ideal $\mathcal{I}_U$ of $U \simeq U \times 0$ in W. Thus, we obtain a complete intersection based relative chart $(j : U \to W, \mathbf{f})$.

Let us show the existence of natural $\mathcal{O}_{U,z}$-isomorphisms:

$$(***) \qquad \mathcal{H}^{p,s}_{W^i/S,\mathbf{f}^i,z_i} \simeq \mathcal{H}^{p,s}_{W/S,\mathbf{f},z}$$

commuting with De-Rham action for $i = 1,2$. We show only the $i = 1$ case (the other case follows similarly). For the end, we explicitly analyze the chain complex $(**)$ (see *Proof* of *Lemma* 4.7 in p.13) in a neighborhood of each point $z \in U$. Let $\underline{z} = (z^0, \cdots, z^n)$ be a local coordinate system of U at z so that $(\underline{z}, \mathbf{f})$ form a coordinate system of W at z. Let $\mathbf{t} = (t^1, \cdots, t^{\dim S})$ be a local coordinate system of S at the image of z, so that the morphism $\Phi_W : W \to S$ is expressed by the coordinates as $\mathbf{t} = \Phi_W(\underline{z}, \mathbf{f}^1, \mathbf{f}^2)$ so that $\Phi_{W_1} = \Phi_W(\underline{z}, \mathbf{f}^1, 0)$, $\Phi_{W_2} = \Phi_W(\underline{z}, 0, \mathbf{f}^2)$ and $\Phi_U = \Phi_W(\underline{z}, 0)$.

The fact that the restriction $\Phi_W|_{W_1} = \Phi_{W_1}$ is submersive over S implies that already a $\dim_{\mathbb{C}} S$-minor of the part of Jacobi matrix of $\Phi_W(\underline{z}, \mathbf{f}^1, \mathbf{f}^2)$ corresponding to the derivations by the coordinates z^j $(j = 0, \cdots, n)$ and $f^1_1, \cdots, f^1_{l_1}$ is invertible (in a neighborhood of z). Then, in the quotient module $\Omega^1_{W/S} = \Omega^1_W / \sum_{i=1}^{\dim_{\mathbb{C}} S} \mathcal{O}_W d\Phi_{W,i}$, the differentials $df^2_1, \cdots, df^2_{l_2}$ of the remaining coordinates $f^2_1, \cdots, f^2_{l_2}$ form part of an $\mathcal{O}_W$-free basis in a neighborhood of z. Then, again shrinking the charts W_i $(i = 1,2)$ and W suitably, we can apply *Lemma* 4.8 repeatedly, and we obtain the $\mathcal{O}_U$-isomorphism $(***)$.

To show the independence of De Rham operator from a choince of basis $\mathbf{f}$, we cannot use the complex $(**)$ (there does not seem to exist a morphism $d_{DR} : (**)^p \to (**)^{p+1}$ which induces the De Rham operator: $\mathcal{H}^{p,s}_{W/S,\mathbf{f}} \to \mathcal{H}^{p+1,s}_{W/S,\mathbf{f}}$). However, *Lemma* 4.8 (3) together with the naturallity of $\iota^\diamond$ (*Lemma* 4.5 ii)) implies the compatibility of the De Rham operation with the isomorphism $(***)$, and, hence, the independence from a choice of basis $\mathbf{f}$ of the De Rham operator on $\mathcal{H}^{\bullet,s}_{W/S,\mathbf{f}}$.

Proof of Lemma 4.7 (2). It is sufficient to show that the stalk of $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ at a point, say z, of U, where Φ_U is submersive, vanishes for $s > 0$. The assumption on the point z means that the Jacobi matrix of Φ_U with respect to the derivations by $z^0, \cdots, z^n$ has a non-vanishing minor at the point $z \in U$. So, in a neighborhood of z in W, the corresponding minor of the Jacobi matrix of Φ_W does not vanish. This means that $df_1, \cdots, df_l$ form a part of $\mathcal{O}_W$-free basis of $\Omega^1_{W/S}$. Then applying *Lemma* 4.8 inductively for a small neighborhood, we reduce to the relative chart of the form $j : U \to U$, and we conclude that $\mathcal{H}^{p,s}_{W/S,\mathbf{f},z}$ is quasi-isomorphic to a single module $\Omega^p_{U/S,z}$ at z. That is, $\mathcal{H}^{p,0}_{W/S,\mathbf{f},z} \simeq \Omega^p_{U/S,z}$ and $\mathcal{H}^{p,s}_{W/S,\mathbf{f},z} = 0$ for $s > 0$.

This completes the proof of *Lemma* 4.7. □

Notation. As a consequence of *Lemma* 4.7, under the assumption that the relative chart $(j, \mathbf{f})$ is a complete intersection, the module $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$, as an $\mathcal{O}_U$-module on U with De Rham differential operator, depends only on the morphism $\Phi_U : U \to S$ but not on $\mathbf{f}$. Therefore, we shall denote the module also by $\mathcal{H}^{p,s}_{\Phi_U}$ (see *Lemma* 4.10).

Remark 4.9. 1. The support of the modules $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}$ for $s>0$ is contained in the critical set C_{Φ_U} (i.e. locally, we have $\mathcal{I}^m_{C_{\Phi_U}}\mathcal{H}^{p,s}_{W/S,\mathbf{f}}=0$ for some positive integer m), does not imply that the module may not be an $\mathcal{O}_{C_{\Phi_U}}$-module.

2. In view of [19], $\mathcal{H}^{p,s}_{W/S,\mathbf{f}}=0$ for $s<\mathrm{depth}(\mathcal{I}_{\Phi_U})$. But we do not use this fact in the present paper.

E) The complex $(\mathcal{H}^{\bullet,s}_{\Phi}, d_{DR})$ on Z.

As an important consequence of **A)-D)**, we introduce complexes $(\mathcal{H}^{\bullet,s}_{\Phi}, d_{DR})$ of $\mathcal{O}_Z$-coherent sheaves for $s\in\mathbb{Z}$.

Lemma 4.10. *Let $\Phi: Z\to S$ be a flat holomorphic map between complex manifolds and let C_Φ be its critical set loci as given in the Main Theorem. For $s\in\mathbb{Z}$, there exists a chain complex $(\mathcal{H}^{\bullet,s}_{\Phi}, d_{DR})$ of $\mathcal{O}_Z$-coherent modules such that, for any based relative chart $(j:U\to W,\mathbf{f})$, there is a natural isomorphism:*

$$(\mathcal{H}^{\bullet,s}_{\Phi}, d_{DR})|_U \simeq (\mathcal{H}^{\bullet,s}_{W/S,\mathbf{f}}, d_{DR}).$$

In particular, this implies

i) *For $s<0$, $\mathcal{H}^{\bullet,s}_{\Phi}=0$.*

ii) *For $s=0$, there is a natural isomorphism:*

$$(\mathcal{H}^{\bullet,0}_{\Phi}, d_{DR}) \quad\simeq\quad (\Omega^{\bullet}_{Z/S}, d_{DR}).$$

iii) *For $s>0$ and $p\in\mathbb{Z}$, we have*

$$\mathrm{Supp}(\mathcal{H}^{p,s}_{\Phi}) \quad\subset\quad C_\Phi.$$

Proof. Let $(j:U\to W,\mathbf{f})$ be any based relative chart, which is a complete intersection. Applying the construction of (4.13), on the open subset U of Z, we obtain a sequence for $s\in\mathbb{Z}$ of complexes $(\mathcal{H}^{\bullet,s}_{W/S,\mathbf{f}}, d_{DR})$ of $\mathcal{O}_Z$-coherent modules equipped with the De Rham operator action. Let $(j':U'\to W',\mathbf{f}')$ be another complete intersection based relative chart, which introduces the complexes $(\mathcal{H}^{\bullet,s}_{W'/S,\mathbf{f}'}, d_{DR})$ on the open set U'. Then *Lemma* 4.7 together with $(***)$ says that, on the intersection $U\cap U'$, they patch each other naturally so that we obtain the complexes of sheaves on $U\cup U'$. Obviously, Z is covered by charts which extends to complete intersection relative charts, there exists a global sheaf $\mathcal{H}^{p,s}_{\Phi}$ on Z together with the action of a De Rham operator as stated. The statement i) follows from the definition (4.13) and the fact $\mathcal{K}^{p,s}_{W/S,\mathbf{f}}=0$ for $s<0$, ii) follows from (4.14), and iii) follows from Lemma 4.7 (2). $\square$

Remark 4.11. As we see, the chain complexes $(\mathcal{H}^{\bullet,s}_{\Phi}, d_{DR})$ themselves are independent of the choices of relative charts. However, for its construction, we have used the relative charts. Can they be constructed without using the relative charts (or, without using Koszul-De Rham algebras)? (See the following Remark 4.12).

Remark 4.12. It is also possible to consider a quotient algebra $\overline{\mathcal{K}}_{W/S}$ of the Koszul-De Rham algebra $\mathcal{K}_{W/S,\mathbf{f}}$ as follows. Namely, suppose the defining ideal $\mathcal{I}_U$ of U in W has the following finite presentation.

$$\oplus\ \mathcal{O}_W^{l_1} \longrightarrow \oplus\ \mathcal{O}_W^{l_0} \longrightarrow \mathcal{I}_U \longrightarrow 0. \tag{4.17}$$

Explicitly, let $f_1,\cdots,f_{l_0}\in\Gamma(W,\mathcal{O}_W)$ be a system generators of $\mathcal{I}_U$ (i.e. the image of the basis of $\oplus\ \mathcal{O}_W^{l_0}$) and let $(g_j^1,\cdots,g_j^{l_0})\in\Gamma(W,\mathcal{O}_W^{l_0})$ $(j=1,\cdots,l_1)$ be a generating system of relations $g_j^1f_1+\cdots+g_j^{l_0}f_{l_0}=0$ (i.e. the image of the basis of $\oplus\ \mathcal{O}_W^{l_1}$). Then, we define

$$\overline{\mathcal{K}}_{W/S} := \Omega^{\bullet}_{W/S}\langle\xi_1,\cdots,\xi_{l_0}\rangle[\eta_1,\cdots,\eta_{l_0}]/\mathcal{I} \tag{4.18}$$

where $\mathcal{I}$ is the both sided ideal generated by the relations (4.4) and

$$\begin{array}{cc} g_j^1\xi_1+\cdots+g_j^{l_0}\xi_{l_0} & (j=1,\cdots,l_1) \\ g_j^1\eta_1+\cdots+g_j^{l_0}\eta_{l_0}+dg_j^1\xi_1+\cdots+dg_j^{l_0}\xi_{l_0} & (j=1,\cdots,l_1). \end{array} \tag{4.19}$$

Then, as the notation indicates, the algebra (4.18) does not depend on a choice of the presentation (4.17) of the ideal $\mathcal{I}_U$. Furthermore, it is not hard to show that all the three structures Koszul differential ∂_K, De Rham differential d_{DR} and the bi-degree structure $\mathcal{K}^{p,s}_{W/S,\mathbf{f}}$ are preserved on the quotient algebra $\overline{\mathcal{K}}^{p,s}_{W/S}$, and that a parallel statement of the functoriality *lemma*s 4.1 and 4.2 hold, too. Then, for each fixed $p \in \mathbb{Z}$, we may also consider the cohomology of the $\partial_{\mathcal{K}}$. The following question has quite likely a positive answer.

Question. Are the cohomology groups of $(\overline{\mathcal{K}}^{p,\star}_{W/S}, \partial_{\mathcal{K}})$ naturally isomorphic to those of $(\mathcal{K}^{p,\star}_{W/S}, \partial_{\mathcal{K}})$ (i.e. to the groups $(\mathcal{H}^{\bullet,s}_{\Phi}, d_{DR})$ $(s \in \mathbb{Z}_{ge0})$)?

Remark 4.13. In the present paper, we use the complexes $\mathcal{H}^{\bullet,s}_{\Phi}$ $(s \in \mathbb{Z}_{\geq 0})$ only as a supporting actor for the proof of the coherence of the relative De Rham cohomology group of Φ (see Case 3. of §5 **D)**). But, for their definition, the condition that $\Phi|_{C_\Phi}$ is a proper map is un-necessary. Therefore, we may expect a wider use of the complexes in future.

5. Step 4: Lifting of Čech cohomology groups

In this section, we give a final step of a proof of the Main Theorem: the coherence of the direct image $\mathbb{R}\Phi_*(\Omega^\bullet_{Z/S}, d_{Z/S})$ in a neighborhood of any point $t \in S$ for a flat map $\Phi : Z \to S$ with a suitable boundary conditions.

We recall that at Fact 2 of **Step 2**, we showed that, for any point $t \in S$, there exists a Stein open neighborhood $S^* \subset S$ of t and a finite system of relative charts $\mathfrak{U} := \{j_k : U_k \to D_k(1) \times S^*\}_{k=0}^{k^*}$ and a real number $0 < r^* < 1$ such that teh following holds:

1) the intersection relative chart j_K for $K \subset \{0, \cdots, k^*\}$ is complete intersection,
2) for any Stein open subset $S' \subset S^*$ and $r^* \leq \forall r \leq 1$, consider the atlas $\mathcal{U}(r, S') := \{U_k(r, S') := j_k^{-1}(D_k(r) \times S')\}_{k=0}^{k^*}$ (3.7) and the manifold $Z(r, S') := \cup_{k=0}^{k^*} U_k(r, S')$ covered by them. Then the direct image $\mathbb{R}\Phi(\Omega^\bullet_{Z(r,S')/S'})$ are isomorphic to each other for r in $r^* \leq r \leq 1$.

The plan of the proof is the following.

A) We express the Hodge to De Rham spectral sequence over any Stein open subset $S' \subset S^*$ in terms of Čech cohomology groups with coefficients in $\Omega^\bullet_{Z/S}$ with respect to the atlas $\mathcal{U}(r, S')$ (3.7).

B) We "lift" the Čech complex to the lifted atlas $\mathfrak{U}(r, S')$ (3.8) of relative charts. To be exact, in order to lift the coefficient to Koszul-De Rham algebra $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$, we need to enhance the atlas to a based lifted atlas $\tilde{\mathfrak{U}}(r, S')$. The existence of such enhancement shown in *Lemma* 5.1 is a quite non-trivial step in the proof.

C) We compare the Čech complex of $\Omega^\bullet_{Z/S}$ with that of $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$ by the morphism π (4.7) and obtain a short exact sequence where the third term is described again by a Čech complex with respect to the atlas $\mathcal{U}(r, S')$ and coefficient in the sheaf $\mathcal{H}^{\bullet,*}$ whose support is contained in the critical set C_Φ.

D) In the long exact sequence of cohomology groups of the above three Čech complexes, two terms (namely, the first and the third) are independent of the radius r. So the cohomology groups of the third, i.e. of $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$, is also independent of r.

E) We apply the Forster-Knorr Lemma (see [7] and also *Lemma* 5.2 of present paper) to the Čech cohomology groups of $\mathcal{K}^{\bullet,\star}_{W/S,\mathbf{f}}$ and see that they give coherent direct image sheaves on a neighborhood S_m of $t \in S$. On the other hand, the third term (the cohomology of $\mathcal{H}^{\bullet,*}$) is already coherent on S_m since C_Φ is proper over S. Thus, the remaining term in the long exact sequence of the cohomologies, that is, the direct images of the relative De Rham complex are also coherent on S_m.

We start the proof now.

A) Čech complex

We consider the Čech chain complex of the relative De Rham complex $\Omega^p_{Z(r,S')/S'}$ with respect to a Stein covering $\mathcal{U}(r,S') := \{U_k(r,S')\}_{k=0}^{k^*}$ (3.7) of $Z(r,S')$ over any Stein open subset $S' \subset S^*$. As usual, the qth cochain module ($q \in \mathbb{Z}$) is given by

$$\check{\mathrm{C}}^q(\mathcal{U}(r,S'),\Omega^p_{Z(r,S')/S'}) := \bigoplus_{\substack{K\subset\{0,\cdots,k^*\}\\ \#K=q+1}} \Gamma(U_K(r,S'),\Omega^p_{Z'(r,S')/S'}). \tag{5.1}$$

(where the summation index K runs also over the cases when $U_K(r,S') = \emptyset$). The Čech coboundary operator is the alternating sum

$$\check{\delta} := \sum_{K\subset K'} \pm(\rho_K^{K'})^* : \ \check{\mathrm{C}}^q(\mathcal{U}(r,S'),\Omega^p_{Z(r,S')/S'}) \to \check{\mathrm{C}}^{q+1}(\mathcal{U}(r,S'),\Omega^p_{Z(r,S')/S'}) \tag{5.2}$$

of pull-back morphisms associated to the inclusion map $\rho_K^{K'} : U_{K'} \to U_K$, where K and $K' \subset \{0,\cdots,k^*\}$ are indices satisfing $\#K = q+1$, $\#K' = q+2$ and $K \subset K'$.

B) Based lifting atlas

Recall the lifting atlas $\mathfrak{U}(r,S') := \{j_k|_{U_k(r,S')} : U_k(r,S') \to D_k(r)\times S'\}_{k=0}^{k^*}$ (3.8) of the atlas $\mathcal{U}(r,S')$ (3.7), where each j_k is a pair (φ_k,Φ) of maps such that φ_k is a local isomorphism of a neighborhood $B(\underline{z}_k, R(\underline{z}))$ of $\underline{z}_k \in Z$ to a domain in $\mathbb{C}^N$. We attach one more structure, i.e. base (recall Definition 4), to the atlas $\mathfrak{U}(r,S')$ as in the following definition.

Definition. 6. A *based lifting atlas* of $\mathcal{U}(r,S')$ is a triplet

$$\widetilde{\mathfrak{U}}(r,S') \ := \ (\mathfrak{U}(r,S'), \mathbf{f}_K, \mathbf{\Pi}_K^{K'})$$

such that

1) $\mathfrak{U}(r,S') = \{j_k \mid 0 \le k \le k^*\}$ is the relative atlas already given in (3.8),

2) $\mathbf{f}_K$ is a minimal generator system of the ideal $\mathcal{I}_{U_K(r,S*)}$ for $K \subset \{0,1,\cdots,k^*\}$. That is, $(j_K,\mathbf{f}_K)$ is a based relative chart in the sense of **Definition 4**.

3) $\mathbf{\Pi}_K^{K'}$ is a based morphism: $(j_{K'},\mathbf{f_{K'}}) \to (j_K,\mathbf{f_K})$ (in the sense of *Lemma* 4.1) for $K,K' \subset \{0,1,\cdots,k^*\}$ with $K \subset K'$ such that

$$\mathbf{\Pi}_K^{K''} = \mathbf{\Pi}_K^{K'} \circ \mathbf{\Pi}_{K'}^{K''}$$

for any $K,K',K'' \subset \{0,1,\cdots,k^*\}$ with $K \subset K' \subset K''$.

We remark that any based reltive chart in a based lifting atlas is automatically a complete intersection in the sense of **Definition 5.** The following existence of based lifting atlases is one crucial step towards the proof of the Main Theorem.

Lemma 5.1. *For the atlas* $\mathfrak{U}(r,S')$*, there exists a based lifting* $\widetilde{\mathfrak{U}}(r,S')$.

Proof. We construct the based lifting explicitly in the following 1), 2) and 3).

1) Recall the notation of the proof of *Lemma* 3.2. For any subset $K \subset \{0,1,\cdots,k^*\}$, we have $j_K : \underline{z}' \in U_K \mapsto ((\varphi_{z_i}(\underline{z}'))_{i=0}^{k^*}, \Phi(\underline{z}')) \in \prod_{i=0}^{k^*} D_{z_i}(r_{z_i}) \times S_K$.

2) As was suggested already by 1) and 2) in the proof of *Lemma* 3.2 (page 6), we choose $\mathbf{f}_K$ as follows.

1. If $U_K = \emptyset$, then we set $\mathbf{f}_K = \{\ 1\ \}$.
2. If $U_K \neq \emptyset$, then $\mathbf{f}_K$ is the union of two parts $\mathbf{f}_{K,I}$ and $\mathbf{f}_{K,II}$ where

$$\mathbf{f}_{K,I} = \left\{z^j \circ \varphi_k^{-1} - z^j \circ \varphi_{k'}^{-1}\right\}_{j=1,k,k'\in K}^{N} \quad \& \quad \mathbf{f}_{K,II} = \left\{t_i - \Phi_i \circ \varphi_{k_0}^{-1}\right\}_{i=1}^{\dim_{\mathbb{C}} S},$$

where k' =the least element of K which is larger than k, and $k_0 = \min\{K\}$.

3) Let $K,K' \subset \{0,1,\cdots,k^*\}$ such that $K \subset K'$. We construct a morphism $\Pi_K^{K'} = (\pi_K^{K'}, h_K^{K'}) : (j_{K'},\mathbf{f}_{K'}) \to (j_K,\mathbf{f}_K)$. As a map $\pi_K^{K'}$ from the relative chart $j_{K'}$ to j_K, we consider the pair consisting of natural projection: $\pi_K^{K'}: D_{K'}(r)\times S' \to D_K(r)\times S'$ and the natural (induced) inclusion: $U_{K'} \to U_K$.

Let us choose and fix a morphism $h_K^{K'}$ between two basis $\mathbf{f}_K$ and $\mathbf{f}_{K'}$.

In case $U_{K'} = \emptyset$, $\mathbf{f}_{K'} = \{1\}$ and we set $h_K^1 = \mathbf{f}_K$.

In case $U_{K'} \neq \emptyset$, then, according to the two groups of basis of $\mathbf{f}_K$ and $\mathbf{f}_{K'}$ in the above 1), we decompose the matrix $h_K^{K'}$ into 4 blocks $\begin{pmatrix} h_{K,I}^{K',I}, & h_{K,I}^{K',II} \\ h_{K,II}^{K',I}, & h_{K,II}^{K',II} \end{pmatrix}$, and fix the morphism blockwise in the following steps 1., 2. and 3.

1. There is a unique way to express any element of $\mathbf{f}_{K,I}$ as a sum of elements of $\mathbf{f}_{K',I}$, then $h_{K,I}^{K',I}$ is its coefficients matrix. Thus we get: $\mathbf{f}_{K,I} = h_{K,I}^{K',I}\mathbf{f}_{K',I}$.

2. We put $h_{K,I}^{K',II} = 0$.

3. We express $\mathbf{f}_{K,II} = h_{K,II}^{K',I}\mathbf{f}_{K',I} + h_{K,II}^{K',II}\mathbf{f}_{K',II}$, where $h_{K,II}^{K',II}$ is the identity matrix of size $\dim_{\mathbb{C}} S$. In order to fix the part $h_{K,II}^{K',I}$, we prepare some functions.

For each i with $1 \leq i \leq \dim_{\mathbb{C}} S$, we express, locally in a Stein coordinate neighborhood, Φ_i (the ith component of the map Φ) as a function $\Phi_i(\underline{z})$ of N variables $\underline{z} = (z^1, \cdots, z^N)$. Consider a copy $\Phi_i(\underline{z}')$ of the function for a coordinate system $\underline{z}' = (z'^1, \cdots, z'^N)$. Then, on the product domain of the coordinate neighborhood, we can find functions $F_{ij}(\underline{z}, \underline{z}')$ $(j = 1, \cdots, N)$ such that

$$\Phi_i(\underline{z}') - \Phi_i(\underline{z}) \quad = \quad \sum_{j=1}^{N} F_{ij}(\underline{z}', \underline{z})(z'^j - z^j), \tag{5.3}$$

since the product domain is Stein where the ideal defining the diagonal is globally generated by $z'^j - z^j$ $(j=1,\cdots,N)$. Then, again taking a copy $\Phi_i(\underline{z}'')$ on the triple product domain and summing up two copies of above formula, we obtain a formula

$$\sum_{j=1}^{N} F_{ij}(\underline{z}'', \underline{z})(z''^j - z^j) = \sum_{j=1}^{N} F_{ij}(\underline{z}', \underline{z})(z'^j - z^j) + \sum_{j=1}^{N} F_{ij}(\underline{z}'', \underline{z})(z''^j - z'^j). \tag{5.4}$$

We return to the construction of the matrix $h_{K,II}^{K',I}$. That is, we need to express the difference: $(t_i - \Phi_i \circ \varphi_{k_0}^{-1}) - (t_i - \Phi_i \circ \varphi_{k_0'}^{-1}) = \Phi_i \circ \varphi_{k_0'}^{-1} - \Phi_i \circ \varphi_{k_0}^{-1}$ as a linear combination of $z^j \circ \varphi_{k_0'}^{-1} - z^j \circ \varphi_{k_0}^{-1}$. The formula (5.3) gives an answer:

$$\Phi_i \circ \varphi_{k_0'}^{-1} - \Phi_i \circ \varphi_{k_0}^{-1} \quad = \quad \sum_{j=1}^{N} F_{ij}(\varphi_{k_0'}^{-1}, \varphi_{k_0}^{-1})(z^j \circ \varphi_{k_0'}^{-1} - z^j \circ \varphi_{k_0}^{-1}),$$

and we obtain the definition: $h_{K,II}^{K',I} = \{F_{ij}(\varphi_{k_0'}^{-1}, \varphi_{k_0}^{-1})\}_{i=1,\cdots,\dim_{\mathbb{C}} S,\ j=1,\cdots,N}$.

Finally, we need to show that the above defined matrix satisfies the functoriality $h_K^{K''} = h_K^{K'} h_{K'}^{K''}$. We can prove this again by decomposing the matrix into 4 blocks, where the cases of the blocks $\binom{I}{I}$, $\binom{II}{I}$ and $\binom{II}{II}$ are trivial. The case of block $\binom{I}{II}$ follows from the addition formula (5.4).

This completes the proof of an existence of based lifting of the atlas $\mathcal{U}(r, S')$. $\square$

Remark 5.2. The above construction does not give a canonical lifting, but depends on the choices of the decomposition (5.3) which is based on rather an abstract existence theorem (cf. [9]). We don't know the meaning of this freedom to the De Rham cohomology group we are studying. As we see in sequel, for the proof of coherence, any choice of the lifting does work. See also Remark 4.11.

From now on, we consider the base lifted atlas $\widetilde{\mathfrak{U}}(r, S')$ for all $r^* \leq r \leq 1$ and Stein open subset $S' \subset S^*$, depending on a choice of functions F_{ij} in (5.3). Since we, later on, want to compare them for different r and S', we first fix the functions F_{ij} and hence a based lifting $(\mathbf{f}_K, \mathbf{\Pi}_K^{K'})$ on the largest atlas $\mathfrak{U}(1, S^*)$, then we consider the induced based lifting to any atlas $\mathfrak{U}(r, S')$.

We lift the Čech (co)chain complex (5.1) to the following triple chain complex. Namely, for $p, q, s \in \mathbb{Z}_{\geq 0}$, we define the cochain module

$$\check{\mathrm{C}}^q(\widetilde{\mathfrak{U}}(r,S'), \mathcal{K}_\Phi^{p,s}) := \underset{\substack{K\subset\{0,\cdots,k^*\}\\ \#K=q+1}}{\oplus} \Gamma(D_K(r)\times S', \mathcal{K}^{p,s}_{D_K(r)\times S'/S',\mathbf{f}_K}).^{13} \tag{5.5}$$

The actions of (co-)boundary operators d_{DR} and $\partial_{\mathcal{K}}$ on the coefficient $\mathcal{K}_\Phi^{\bullet,\star}$ preserve the chart, so that they induce a double complex structure $(\check{\mathrm{C}}^q(\widetilde{\mathfrak{U}}(r,S'), \mathcal{K}_\Phi^{\bullet,\star}), d_{DR}, d_{\mathcal{K}})$. We now lift the Čech coboundary operator on (5.1) to the lifted module (5.5).

For $p, q, s \in \mathbb{Z}$, we introduce an $\Gamma(S', \mathcal{O}_S)$-*homomorphism*

$$\check{\delta} := \sum_{K\subset K'} \pm(\Pi_K^{K'})^\diamond \ : \ \check{\mathrm{C}}^q(\widetilde{\mathfrak{U}}(r,S'), \mathcal{K}_\Phi^{p,s}) \longrightarrow \check{\mathrm{C}}^{q+1}(\widetilde{\mathfrak{U}}(r,S'), \mathcal{K}_\Phi^{p,s}), \tag{5.6}$$

where $(\Pi_K^{K'})^\diamond$ is the pull-back morphism (4.6) in *Lemma* 4.1 associated with the morphism $\Pi_K^{K'} = (\pi_K^{K'}, h_K^{K'})$ given in 3) of the proof of *Lemma* 5.1, and the sign and the running index K and K' are the same as those for the Čech coboundary operator (5.2). We shall call this morphism the *lifted* Č*ech coboundary operator*. The lifted Čech coboundary operator satisfies the relations:

$$\check{\delta}^2 = 0, \quad \check{\delta} d_{DR} + d_{DR}\check{\delta} = 0, \quad \text{and} \quad \check{\delta}\partial_{\mathcal{K}} + \partial_{\mathcal{K}}\check{\delta} = 0.$$

Proof. To show that $\check{\delta}^2 = 0$ is the same calculation as the standard Čech coboundary case. Other relations follow from the fact that the pull-back homomorphism $(\pi_K^{K'})^\diamond$ commutes with d_{DR} and $\partial_{\mathcal{K}}$ (*Lemma* 4.1). □

C) Comparison of the triple Čech-complex of $\mathcal{K}_\Phi^{\bullet,\star}$ with the double Čech-complex of $\Omega_\Phi^\bullet$

We compare the triple complex (5.5) with the double complex (5.1). More exactly, for our restricted purpose (to calculate the second page of the Hodge to De Rham spectral sequence), we fix the index p for the chain complex for De Rham differential operator. That is, we compare only the remaining double complex of the two coboundary operators $(\check{\delta}, \partial_{\mathcal{K}})$ with the Čech (co)chain complex of the coboundary operator $\check{\delta}$. The comparison is achieved by the morphism π (recall §4 **B)**).

$$\cdots \xrightarrow{\partial_{\mathcal{K}}} \check{\mathrm{C}}^{\,q}(\widetilde{\mathfrak{U}}(r,S'), \mathcal{K}_\Phi^{p,1}) \xrightarrow{\partial_{\mathcal{K}}} \check{\mathrm{C}}^{\,q}(\widetilde{\mathfrak{U}}(r,S'), \mathcal{K}_\Phi^{p,0}) \xrightarrow{\pi} \check{\mathrm{C}}^{\,q}(\mathcal{U}, \Omega^p_{Z(r,S')/S'}) \to 0.$$

The commutativity of π with the lifted and un-lifted Čech coboundary operator is, termwise, equivalent to the commutativity $\rho^{U_K}_{U_{K'}} \circ \pi = \pi' \circ (\pi_K^{K'})^\diamond$ (4.9).

Let us consider the total complex of (5.5) with respect to $\check{\delta}$ and $\partial_{\mathcal{K}}$ by putting $\tilde{*} := * - \star$ and $\tilde{\partial} := \check{\delta} + \partial_{\mathcal{K}}$:

$$(Tot^{\tilde{*}}\check{\mathrm{C}}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'), \mathcal{K}_\Phi^{p,\cdot}), \tilde{\partial}) \tag{5.7}$$

where

$$Tot^{\tilde{*}}\check{\mathrm{C}}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'), \mathcal{K}_\Phi^{p,\cdot}) := \underset{*-\star=\tilde{*}}{\oplus} \check{\mathrm{C}}^{*}(\widetilde{\mathfrak{U}}(r,S'), \mathcal{K}_\Phi^{p,\star}). \tag{5.8}$$

In view of (4.7), for each fixed $p \in \mathbb{Z}$, the chain morphism

$$(Tot^{\tilde{*}}\check{\mathrm{C}}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'), \mathcal{K}_\Phi^{p,\cdot}), \tilde{\partial}) \ \xrightarrow{\pi} \ (\check{\mathrm{C}}^{*}(\mathcal{U}, \Omega^p_{Z(r,S')/S'}), \check{\delta}) \tag{5.9}$$

[13] In the notation of LHS, we replaced the subscript like $D_K(r)\times S/S$ indicating where the module is defined by Φ, since we may regard $\mathcal{K}_\Phi^{p,s}$ to be a sheaf satisfying the functoriality (*Lemma* 4.1) defined on all relative charts, depending on the choice of a based lifting in *Lemma* 5.1.

is an epimorphism in the category of cochain complexes. So, using the kernel of it, we obtain a short exact sequence:

$$
\begin{aligned}
0 \to (Tot^{\tilde{*}}\check{\mathrm{C}}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}^{p,\cdot}_{\Phi,\ker(\pi)}),\tilde{\partial}) &\xrightarrow{\iota} (Tot^{\tilde{*}}\check{\mathrm{C}}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}^{p,\cdot}_{\Phi}),\tilde{\partial}) \\
&\xrightarrow{\pi} (\check{\mathrm{C}}^{*}(\mathcal{U},\Omega^{p}_{Z(r,S')/S'}),\check{\delta}) \;\to\; 0,
\end{aligned} \tag{5.10}
$$

where the kernel (the first term) is again the total complex of a lifted Čech chain complex of the atlas $\widetilde{\mathfrak{U}}(r,S')$ with coefficients in a complex $\mathcal{K}^{p,s}_{\Phi,\ker(\pi)}$ (for fixed p), which is the sub-complex of $\mathcal{K}^{p,s}_{\Phi}$ obtainded by replacing the first term $\mathcal{K}^{p,0}_{\Phi}$ by the term $\partial_{\mathcal{K}}(\mathcal{K}^{p,1}_{\Phi}) = \ker(\pi : \mathcal{K}^{p,0}_{\Phi} \to \Omega^{p}_{\Phi})$, and ι is the map induced from the natural inclusion $\mathcal{K}^{p,s}_{\Phi,\ker(\pi)} \subset \mathcal{K}^{p,s}_{\Phi}$.

Due to the commutativity of π with the De Rham differential operator (4.8), the chain maps (5.9) commute with De Rham operator action between the modules for the indices p and $p+1$. That is, by taking the direct sum over the index $p \in \mathbb{Z}$, we may regard π as an epimorphism from the double complex of $(\tilde{\partial}, d_{DR})$ to the double complex of $(\check{\delta}, d_{Z/S})$. Then, similarly, by taking the direct sum of the sequences (5.10) over the index $p \in \mathbb{Z}$, we obtain a short exact sequence of double complexes.

Before calculating cohomology long exact sequence of the short exact sequence, we show in the following lemma some *finiteness* and *boundedness* of the total complex (5.7) (considered as a double complex of the indices $\tilde{*}$ and p), which makes big contrast with the case of *Lemma* 4.2. Namely, in case of the total complex of $\partial_{\mathcal{K}}$ and d_{DR}, we did not get such finiteness and boundedness (see Remark 4.4). This finiteness, which holds for the total complex of ∂_K and $\check{\delta}$, is one of the most subtle but the key point where Koszul-De Rham algebra works mysteriously.

Lemma 5.3. *The complex* (5.7) *is finite and bounded in the following two senses.*

i) The RHS of (5.8) *for fixed p and* $\tilde{*}$ *is a finite direct sum of the form*

$$
\bigoplus_{q=-1}^{k^*-1} \check{C}^{q}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}^{p,q-\tilde{*}}_{\Phi}) = \bigoplus_{K\subset\{0,\cdots,k^*\}} \Gamma(D_K(r)\times S', K^{p,\#K-\tilde{*}-1}_{D_K(r)\times S'/S',\mathbf{f}_K}).
$$

ii) The set $\{(p,\tilde{*}) \in \mathbb{Z}^2 \mid Tot^{\tilde{*}}\check{C}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}^{p,\cdot}_{\Phi}) \neq 0\}$ *is contained in a strip*

$$
-(k^*+1)(N-1)+n-1 \;\le\; \tilde{*}+p \;\le\; (k^*+1)(N+1)-1. \tag{5.11}
$$

Proof. i) The summation index $K \subset \{0,\cdots,k^*\}$ (5.5) runs over a finite set so that $* = \#K-1$ is bounded. Then the condition that $* - \star = \tilde{*}$ is fixed means that the range of $\star$ is bounded.

ii) Recall $Tot^{\tilde{*}}\check{\mathrm{C}}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}^{p,\cdot}_{\Phi}) := \bigoplus_{*-\star=\tilde{*}} \bigoplus_{\#K=*+1} \Gamma(D_K(r)\times S', \mathcal{K}^{p,\star}_{D_K(r)\times S'/S'})$. If there is a non-vanishing term in RHS for some $*$, $\star$ and K, then due to (4.11), one has $-l_K \le p-\star \le \dim_{\mathbb{C}} D_K(r)$. Then, adding $* = \star + \tilde{*}$ in both hand sides, we have $-l_K + * \le \tilde{*}+p \le \dim_{\mathbb{C}} D_K(r) + *$. Since $* = \#K-1$ and $l_K = \#K\cdot\dim_{\mathbb{C}} S + (\#K-1)n$, $\dim_{\mathbb{C}} D_K(r) = \#K \cdot N$ (recall §3) and $\dim_{\mathbb{C}} Z = N = m = n + \dim_{\mathbb{C}} S$, we get

$$
-\#K(N-1)+n-1 \;\le\; \tilde{*}+p \;\le\; \#K(N+1)-1
$$

Since the index K runs over all subsets of $\{0,1,\ldots,k^*\}$, we obtain the formula. □

According to i) and ii) of *Lemma* 5.3, we have two important consequences: i) the cohomology groups is described by a finite chain complex where each chain module is a finite direct sum of the spaces of holomorphic functions on some relative charts (this descriptions is necessary to apply the Forster-Knorr Lemma), and ii) for each

fixed p, the complex is bounded. This observation leads us to introduce the following truncation of the double complexes.

$$
(5.12)\qquad
\begin{aligned}
TR^{p,\tilde{*}} &:= \begin{cases} Tot^{\tilde{*}}\check{C}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_{\Phi}^{p,\cdot}) & \text{if } 0\le p\le \dim_{\mathbb{C}} Z\\ 0 & \text{otherwise}\end{cases}\\
TR^{p,\tilde{*}}_{\ker(\pi)} &:= \begin{cases} Tot^{\tilde{*}}\check{C}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}_{\Phi,\ker(\pi)}^{p,\cdot}) & \text{if } 0\le p\le \dim_{\mathbb{C}} Z\\ 0 & \text{otherwise}\end{cases}
\end{aligned}
$$

For the truncated double complexes, we have

i) The complexes $TR^{p,\tilde{*}}$ and $TR^{p,\tilde{*}}_{\ker(\pi)}$ are bounded for the both indices p and $\tilde{*}$, and also from above and below.

ii) The following is an exact sequence of bounded double complexes:

(5.13)

$$0\to(TR^{p,\tilde{*}}_{\ker(\pi)},d_{DR},\tilde{\partial})\xrightarrow{\iota}(TR^{p,\tilde{*}},d_{DR},\tilde{\partial})\xrightarrow{\pi}(\check{C}^{*}(\mathcal{U},\Omega^{p}_{Z(r,S')/S'}),d_{Z/W},\check{\delta})\to 0,$$

which is the goal of our construction. From now on, we start to analyze the sequence.

D) Long exact seqence of images on S.

We consider now the long exact sequence of the cohomology group associated to the short exact sequence obtained from (5.13) by taking the total complex for each of the three double complexes. Recalling the construction of the atlases $\mathcal{U}(r,S')$ (3.7) and $\mathfrak{U}(r,S')$ (3.8), we note that each term of the sequence depends on the choice of a set S' and a real number r with $r^*\le r\le 1$. By fixing r and running S' over all Stein open subset of S^*, we obtain a sheaf on S^* (depending on r).

In the following, we analyze the module (sheaf or its sections over S') of the cohomology groups with the three coefficients cases separately.

Case 1. $(\check{C}^{*}(\mathcal{U},\Omega^{p}_{Z(r,S')/S'}),d_{Z/W},\check{\delta})$.

The module is exactly the module of relative De Rham hyper-cohomology group $\mathbb{R}\Phi_*(\Omega^{\bullet}_{Z_{S^*}/S^*},d_{Z_{S^*}/S^*})$ (3.6) of the morphism $\Phi: Z(r)\to S^*$. It is shown that the module is independent of the choice of r with $r^*\le r\le 1$.

Case 2. $(TR^{p,\tilde{*}},d_{DR},\tilde{\partial})$.

Due to the finiteness *Lemma* 5.3 i) and the boundedness of the double complex, the cohomology group is expressed as a cohomology group of a finite complex, where each chain module is a finite direct sum of a module of the form $\Gamma(D(r)\times S',\mathcal{O}_{D(r)\times S'})$ for some polydisc $D(r)$ of radius r.

Proof. Recall the direct sum decomposition (4.10). Noting that W/S is given by $D_K(r)\times S'/S'$ and we obtain $\Omega^p_{W/S}=\oplus_{i_1<\cdots<i_p}\mathcal{O}_{D_K(r)\times S'}dz_{i_1}\wedge\cdots\wedge dz_{i_p}$ for a coordinate system $\underline{z}$ of the polydisc $D_K(r)$. □

Case 3. $(TR^{p,\tilde{*}}_{\ker(\pi)},d_{DR},\tilde{\partial})$.

We approach the cohomology group of this case by a use of the spectral sequence of the double complex w.r.t. d_{DR} and $\tilde{\partial}$. Let us first calculate the cohomology group of the double complex with respect to the coboundary operator $\tilde{\partial}$ (in order to avoid a confusion, let us call the spectral sequence E_I). Thus, each entry of the first page of the spectral sequence E_I is again the total cohomology group of the total complex $(Tot^{\tilde{*}}\check{C}^{\cdot}(\widetilde{\mathfrak{U}}(r,S'),\mathcal{K}^{\bullet,\cdot}_{\Phi,\ker(\pi)}),\tilde{\partial}=\check{\delta}+\partial_K)$. Again we approach the group from the spectral sequence of the double complex w.r.t. $\check{\delta}$ and ∂_K.

Let us first consider the spectral sequence obtained by considering the cohomology group with respect to the coboundary operator ∂_K first, for the reason below (let us call this spectral sequence E_{II}).

Recall *Lemma* 4.10 that it was shown that there exists a sequence of complexes $\mathcal{H}^{\bullet,s}_{\Phi}$ $(s\in\mathbb{Z})$ of coherent $\mathcal{O}_Z$–modules such that 1) the restriction of the s-th complex to U_W induces a natural isomorphism to the s-th cohomology group of the Koszul-De Rham double complex $(\mathcal{K}^{\bullet,\star}_{W/S},d_{DR},\partial_K)$ with respect to the coboundary operator

∂_K, and 2) the support of the module for $s > 0$ is contained in the critical set C_Φ. Thus, the (q,s)-entries of E_{II} is given by direct images $\check{\mathrm{C}}^q(\widetilde{\mathfrak{U}}(r,S'),\mathcal{H}_\Phi^{\bullet,s})$ of coherent sheaves $\mathcal{H}_\Phi^{\bullet,s}$ (the fact that the pair d_{DR} and $\check{\delta}$ forms a double complex structure on $\oplus_{p,q}\check{\mathrm{C}}^q(\mathcal{U}(r,S'),\mathcal{H}_\Phi^{p,s})$ is verified by a routine). In view of the fact that $C_\Phi \subset Z'$ is proper over the base space S, this, in particular, implies that 1) the entry is independent of r, and 2) the sheaf obtained by running S' over all Stein open subset of S^* is an $\mathcal{O}_{S^*}$-coherent module. Then, these two properties should be inherited by the limit of the spectral sequence E_{II} and the associated total cohomology group.

Coming back to the spectral sequence E_I, we see that all the entries of the first page of E_I have the above properties 1) and 2). Thus the cohomology group of the total complex of the double complex $(TR^{p,\tilde{*}}_{\ker(\pi)}, d_{DR}, \tilde{\partial})$ should have the property. Then in view of the long exact sequence, we started, two terms Case 1. and 3. of them (as a triangle) are independent of r. Thus, we conclude that the third term Case 2. satisfies:

The total complex of the double complex $(TR^{p,\tilde{*}}, d_{DR}, \tilde{\partial})$ is quasi-isomorphic to each other for r and r' with $r^* \le r, r' \le 1$.

E) Application of the Forster-Knorr Lemma.

We are now able to apply the following key Lemma due to Forster and Knorr [7] [15] (the formulation here of the result is taken from their unpublished note which is slightly modified from the published one, however can be deduced).

Lemma 5.4. (Forster-Knorr) *Let m be a given integer, S a smooth complex manifold, 0 a point in S. Suppose that $(C^*(r), d)$ is a complex of $\mathcal{O}_S$-modules bounded from the left such that*

i) for any Stein open subset $S' \subset S$ and $q \in \mathbb{Z}$, we have an isomorphism

$$C^q(r)(S') \simeq \prod_{finite} \Gamma(D(r) \times S', \mathcal{O}_{D(r)\times S'})$$

together with the Fréchet topology. Here, $D(r)$ is a polycylinder of radius $r \in \mathbb{R}_{>0}$ whose dimension varies depending on each factor.

ii) $d : C^q(r) \to C^{q+1}(r)$ is an $\mathcal{O}_S$-homomorphism, which is continuous with respect to the Fréchet topology.

iii) There exist r_1 and r_2 such that, for any r, $r_1 \ge r \ge r' \ge r_2 > 0$, the restriction $C^*(r) \to C^*(r')$ is a quasi-isomorphism.

Then, there exists a small neighborhood S_m of 0 in S (depending on $m \in \mathbb{Z}$) such that , for $q \ge m$, $\mathrm{H}^q(C^*(r))\big|_{S_m}$ is an $\mathcal{O}_{S_m}$-coherent module.

We apply *Lemma* 5.4 to the total complex of complex $(TR^{p,\tilde{*}}, d_{DR}, \tilde{\partial})$.

Let us check that the complex satisfies the assumptions in the Forster-Knorr Lemma by putting $S = S^*$ (and run S' over all Stein open subset of S^* in order to make $\mathcal{O}_{S^*}$-module strcture), 0 to be $t \in S^*$ and $r_1 = 1,\ r_2 = r^*$.

i) The condition i) is satisfied due to the description in **D), Case** $(TR^{p,\tilde{*}}, d_{DR}, \tilde{\partial})$.

ii) The condition ii) is verified as follows. The coboundary operator here is a mixture of $\partial_K, \check{\delta}$ and d_{DR}, all of them are obviously $\mathcal{O}_S$-homomorphisms. That they are continuous w.r.t. the Fréchet topology can be seen as follows.

It is well known that the holomorphic function ring (Stein algebra) $\Gamma(D_K(r) \times S', \mathcal{O}_{D_K(1)\times S^*})$ carries naturally a Fréchet topology ([3], [9] p266) with respect to the compact open convergence. Then, the operators $\check{\delta}$ and $\partial_{\mathcal{K}}$ are $\mathcal{O}_{D_K(1)\times S^*}$-homomorphisms and induce continuous morphisms on the modules. The operator d_{DR} is no longer an $\mathcal{O}_{D_K(1)\times S^*}$-homomorphism but is only an $\mathcal{O}_{S^*}$-homomorphism. Nevertheless, it is also well known that differentiation operators on a Stein algebra are also continuous w.r.t. the Fréchet topology.

iii) The quasi-isomorphisms between the complexes for r and r' with $r* \leq r, r' \leq 1$ was shown in the last step of **D)**.

Finally, choosing $m = -1$, we obtain the coherence of the direct image sheaf of the total complex of $(TR^{p,\tilde{*}}, d_{DR}, \tilde{\partial})$ in a neighborhood of $t \in S^*$. Then, we return to the long exact sequence studied in **D)**. Two terms Case 2. and 3. of them (as a triangle) are $\mathcal{O}_S$-coherent near at $t \in S$. Therefore, the third term Case 3., the direct image of the double complex, that is, *the hyper-cohomology groups* $\mathbb{R}\Phi_*(\Omega^\bullet_{Z/S}, d_{Z/S})$ *is also* $\mathcal{O}_S$*-coherent in a neighborhood of* $t \in S$.

This completes the proof of the **Main Theorem** given in Introduction. □

References

[1] D. Burghelea and M.V. Poirrier, *Cyclic homology of commutative algebras I* , Algebraic topologyrational homotopy (Louvain-la-Neuve, 1986), Lecture Notes in Math. 1318, Springer Verlag, Berlin, (1988), 51-72.

[2] E. Brieskorn, *Die Monodromie der isolierten Singularitäten von Hyperflächen*, Manuscripta Math. **2** (1970), 103–161.

[3] H. Cartan, Séminaire Cartan, Éc. norm. Supér. '51-52

[4] P. Deligne, *Théorie de Hodge. II. Publ. Math. IHES*, 1971, **40**, 5-58.

[5] P. Deligne, *Théorie de Hodge. III. Publ. Math. IHES*, 1974, **44**, 5-77.

[6] G. De Rham, Sur la division de formes et de courants par une forme linéaire, Comment. math. Helv. 28, (1954), 346-352.

[7] O. Forster and K. Knorr, *Ein Beweis des Grauertschen Bildgarbensatzes nach Ideen von B. Malgrange*, Manuscripta Math. **5** (1971), 19-44.

[8] H. Grauert, *Ein Theorem der Analytischen Garbenthoerie und die Modulräume komplexer Strukturen*, Publ. Math. I.H.E.S. 5, Paris 1960.

[9] H. Grauert, Th. Peternell and R. Remmert, *Several Complex variables VII*, encyclopedia of Mathematical Sciences, Volume 74

[10] G. M. Greuel, *Der Gauss-Manin Zusammenhang isolierter Singularitäten von vollständigen Durchschnitten, Math. Ann.* (1975), **214**, 235-66.

[11] H. Hamm, *Zur analytischen und algebraischen Beschreibung der Picard-Lefschetz-Monodromie,* Habilitationsschrift, Göttingen, (1974), 106pp.

[12] C. Houzel and P. Schapira *Images diretes de modules différentiels,* C. R. Acad. Sc. Paris, t. 298, Série I, n° 18, 1984.

[13] N. Katz, *The regularity theorem in Algebraic Geometry*, Proceedings of I.C.M., Nice, 1970.

[14] N. Katz and T. Oda, *On the differentiation of De Rham cohomology classes with respect to parameters*, J. Math. Kyoto Univ. **8** (1968), 199-213.

[15] K. Knorr, *Der Grauertsche Projektionssatz*, Inventiones math. **12**, (1971), 118-172.

[16] C. Li, S. Li and K. Saito, *Primitive Forms via Poly-vector Fields*, preprint at arXive: math.AG/1311.1659.

[17] K. Saito, *Calcul algébrique de la monodromie*, Société Mathématique de France, Astérisque, 7 et 8, (1973), 195-212.

[18] ———, *Regularity of Gauss-Manin connection of a flat family of isolated singularities*, Quelques journ'ees singulières,École Polytechnique, Paris (1973).

[19] ———, *On a generalization of De Rham Lemma*, Ann. Inst Fourier, Grenoble **26**, **2** (1976), 165-170.

[20] P. Schapira and J-P. Schneiders, *Index Theorem for Elliptic Pairs*, Astérisque **224**, (1994).

[21] R. Thom, Ensembles et Morphismes Stratifiés, Bulletin of AMS 75 (1969), 240-284.

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